

Λ -ADIC GROSS–ZAGIER FORMULA FOR ELLIPTIC CURVES AT SUPERSINGULAR PRIMES

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ABSTRACT. In this paper, we prove a Λ -adic extension of Kobayashi’s p -adic Gross–Zagier formula for elliptic curves at supersingular primes [Kob13]. The main formula is in terms of plus/minus Heegner points over the anticyclotomic tower, and its proof is via Iwasawa theory, based on the connection between Heegner points, Beilinson–Flach elements, and their explicit reciprocity laws. As a key step in the argument, we formulate and prove an analog of Perrin–Riou’s Heegner point main conjecture [PR87a] in this setting, and use this result to complete the proof (under mild hypotheses) of various related one- and two-variable main conjectures.

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INTRODUCTION

Iwasawa theory for elliptic curves at supersingular primes. Let E/\mathbf{Q} be an elliptic curve of conductor of N , let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ be the associated newform, and fix a prime $p \nmid 6N$ of good reduction for E . Assume that E has supersingular reduction at p , i.e., $p|a_p$.

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Since $p \geq 5$, this forces $a_p = 0$ by the Hasse bounds. Let \mathbf{Q}_∞ be the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q} , and denote by Γ the Galois group $\text{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$.

Since its early development in the 1970s, the Iwasawa theory of E over \mathbf{Q}_∞ for supersingular primes posed more challenges than in the ordinary case (where $p \nmid a_p$). Indeed, on the analytic side, the p -adic L -functions

$$L_{p,\alpha}(f/\mathbf{Q}) \in \mathbf{Q}_p[[\Gamma]]$$

constructed by Amice–Vélu and Višik (cf. [MTT86]) for each root α of the Hecke polynomial $x^2 - a_p x + p$ have *unbounded* coefficients, while on the algebraic side, the natural p^∞ -torsion Selmer group $\text{Sel}_{p^\infty}(f/\mathbf{Q}_\infty)$ up the cyclotomic tower is not $\mathbf{Z}_p[[\Gamma]]$ -cotorsion.

However, the situation changed drastically in the early 2000s. On the one hand, in [Pol03] Pollack relegated the study of the unbounded $L_{p,\alpha}(f/\mathbf{Q})$ to the study of two bounded Iwasawa functions $L_p^\pm(f/\mathbf{Q}) \in \mathbf{Z}_p[[\Gamma]]$ for which he showed the decomposition

$$(0.1) \quad L_{p,\alpha}(f/\mathbf{Q}) = L_p^+(f/\mathbf{Q}) \cdot \log_p^- + L_p^-(f/\mathbf{Q}) \cdot \log_p^+ \cdot \alpha$$

for certain ‘half-logarithms’ $\log_p^\pm \in \mathbf{Q}_p[[\Gamma]]$ accounting for the unboundedness of $L_{p,\alpha}(f/\mathbf{Q})$. On the other hand, in [Kob03] Kobayashi defined two plus/minus Selmer groups $\text{Sel}_{p^\infty}^\pm(f/\mathbf{Q}_\infty)$, defined by imposing a more stringent local condition at p than the one defining $\text{Sel}_{p^\infty}(f/\mathbf{Q}_\infty)$, he showed that these new Selmer groups are $\mathbf{Z}_p[[\Gamma]]$ -cotorsion, and conjectured that their characteristic ideals should be generated by Pollack’s plus/minus p -adic L -functions. (In *loc.cit.*, Kobayashi also showed the equivalence between his plus/minus main conjectures and the main conjectures formulated by Kato [Kat93] and by Perrin-Riou [PR93], and deduced from Kato’s work [Kat04] one of the divisibilities predicted by his conjectures.)

Two-variable main conjectures over imaginary quadratic fields. In the following, we fix a root α of $x^2 - a_p x + p$, and let $\beta = -\alpha$ be the other root. Let K/\mathbf{Q} be an imaginary quadratic field of discriminant prime to N , let $\Gamma_K := \text{Gal}(K_\infty/K)$ be the Galois group of the unique \mathbf{Z}_p^2 -extension of K_∞/K unramified outside p , and assume that

$$(spl) \quad p = \mathfrak{p}\bar{\mathfrak{p}} \quad \text{splits in } K.$$

Building on Haran’s construction [Har87] of Mazur–Tate elements for $\text{GL}_{2/K}$, Loeffler has introduced in [Loe13]:

- Four unbounded distributions on Γ_K :

$$(0.2) \quad L_{p,(\alpha,\alpha)}(f/K), \quad L_{p,(\alpha,\beta)}(f/K), \quad L_{p,(\beta,\alpha)}(f/K), \quad L_{p,(\beta,\beta)}(f/K),$$

interpolating the Rankin–Selberg L -values $L(f/K, \psi, 1)$, as ψ runs over the finite order characters of Γ_K ;

- Four bounded measures \mathbf{Q}_p -valued on Γ_K :

$$L_p^{+,+}(f/K), \quad L_p^{-,+}(f/K), \quad L_p^{+,-}(f/K), \quad L_p^{-,-}(f/K),$$

for which one has the decomposition

$$(0.3) \quad \begin{aligned} L_{p,(\alpha,\beta)}(f/K) &= L_p^{+,+}(f/K) \cdot \log_p^+ \log_{\bar{\mathfrak{p}}}^+ \\ &\quad + L_p^{-,+}(f/K) \cdot \log_p^+ \log_{\bar{\mathfrak{p}}}^- \cdot \alpha + L_p^{+,-}(f/K) \cdot \log_p^- \log_{\bar{\mathfrak{p}}}^+ \cdot \beta \\ &\quad + L_p^{-,-}(f/K) \cdot \log_p^- \log_{\bar{\mathfrak{p}}}^- \cdot \alpha\beta, \end{aligned}$$

and similarly for the other three distributions in (0.2), where $\log_p^\pm, \log_{\bar{\mathfrak{p}}}^\pm \in \mathbf{Q}_p[[\Gamma_K]]$ correspond to Pollack’s \log_p^\pm under the identifications $\mathcal{O}_{K,\mathfrak{p}}^\times \simeq \mathbf{Z}_p^\times \simeq \mathcal{O}_{K,\bar{\mathfrak{p}}}^\times$.

On the algebraic side, B.-D. Kim has introduced in [Kim14a] four different doubly-signed Selmer groups $\text{Sel}_{p^\infty}^{\pm, \pm}(f/K_\infty)$ defined by imposing analogues of Kobayashi's plus/minus local conditions at the primes above \mathfrak{p} and $\bar{\mathfrak{p}}$. In contrast to the usual Selmer group, one expects that $\text{Sel}_{p^\infty}^{\pm, \pm}(f/K_\infty)$ is $\mathbf{Z}_p[[\Gamma_K]]$ -cotorsion; in fact, one has the following (cf. [Kim14a, Conj. 3.1]):

Conjecture A. *For each pair of signs $*, \circ \in \{+, -\}$, the Pontryagin dual $\text{Sel}_{p^\infty}^{*, \circ}(f/K_\infty)^\vee$ is $\mathbf{Z}_p[[\Gamma_K]]$ -torsion, and*

$$Ch_{\mathbf{Z}_p[[\Gamma_K]]}(\text{Sel}_{p^\infty}^{*, \circ}(f/K_\infty)^\vee) = (L_p^{*, \circ}(f/K))$$

as ideals in $\mathbf{Z}_p[[\Gamma_K]]$.

In [Wan15], the second author has obtained (under mild hypotheses) one of the divisibilities predicted by the two equal-sign cases of Conjecture A when the global root number of E/K is $+1$. Combined with Kobayashi's work [Kob03], this should yield the predicted equality in those cases by restriction to the cyclotomic line. In this paper, in the course of proving the Λ -adic Gross–Zagier formula in the title, we will similarly deduce the equal-sign cases of Conjecture A when the global root number of E/K is -1 .

Λ-adic Gross–Zagier formula for supersingular primes. Assume from now on that N is square-free and that the imaginary quadratic field K satisfies the following *Heegner hypothesis* relative to N :

(Heeg) N has an *even* number of factors inert in K .

We may decompose $\Gamma_K \simeq \Gamma^{\text{cyc}} \times \Gamma^{\text{ac}}$, where Γ^{cyc} (resp. $\Gamma^{\text{ac}} = \text{Gal}(K_\infty^{\text{ac}}/K)$) is the Galois group of the cyclotomic (resp. anticyclotomic) \mathbf{Z}_p -extension of K , and set $\Lambda^{\text{ac}} := \mathbf{Z}_p[[\Gamma^{\text{ac}}]]$. Then $L_{p,(\alpha,\alpha)}(f/K)$ satisfies a functional equation forcing its vanishing along the ‘line’ consisting of characters of Γ_K factoring through Γ^{ac} . In other words, letting $\gamma^{\text{cyc}} \in \Gamma^{\text{cyc}}$ be a topological generator, and expanding $L_{p,(\alpha,\alpha)}(f/K)$ as a power series in $(\gamma^{\text{cyc}} - 1)$:

$$(0.4) \quad L_{p,(\alpha,\alpha)}(f/K) = L_{p,(\alpha,\alpha),0}(f/K) + L_{p,(\alpha,\alpha),1}^{\text{cyc}}(f/K)(\gamma^{\text{cyc}} - 1) + \cdots,$$

the constant term $L_{p,(\alpha,\alpha),0}(f/K)$ vanishes identically. Via the decomposition (0.3), the functional equation for $L_{p,(\alpha,\alpha)}(f/K)$ gives rise to an analogous functional equation for its components $L_p^{\pm, \pm}(f/K)$, and hence the constant term $L_{p,0}^{\pm, \pm}(f/K) \in \Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ in the expansion

$$(0.5) \quad L_p^{\pm, \pm}(f/K) = L_{p,0}^{\pm, \pm}(f/K) + L_{p,1}^{\pm, \pm}(f/K)(\gamma^{\text{cyc}} - 1) + \cdots$$

vanishes identically as well (see Corollary 1.6). Notice that $L_{p,0}^{\pm, \pm}(f/K)$ is just the restriction of $L_p^{\pm, \pm}(f/K)$ to the anticyclotomic line, and so (in light of Conjecture A and control theorems) the Selmer groups $\text{Sel}_{p^\infty}^{\pm, \pm}(f/K_\infty^{\text{ac}})$ may not be Λ^{ac} -cotorsion.

Next, similarly as in the work of Darmon–Iovita [DI08] (but working with torsion-free rather than torsion coefficients), we construct bounded cohomology classes

$$\mathbf{z}^\pm \in \text{Sel}^{\pm, \pm}(K_\infty^{\text{ac}}, T)$$

using Heegner points over ring class fields of p -power conductor. On the other hand, in [Kim07] B.-D. Kim established the self-duality of plus/minus local conditions under local Tate duality. Building on this, we can deduce from Howard's work [How04a] the existence of Λ^{ac} -adic height pairings

$$(0.6) \quad \langle \cdot, \cdot \rangle_{K_\infty^{\text{ac}}}^{\text{cyc}} : \text{Sel}^{\pm, \pm}(K_\infty^{\text{ac}}, T) \times \text{Sel}^{\pm, \pm}(K_\infty^{\text{ac}}, T) \longrightarrow \Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

Our Λ -adic Gross–Zagier formula is then the following.

Theorem B. *Assume that N is square-free, $p \nmid 6N$ splits in K , hypothesis (Heeg) holds, $E[p]$ is ramified every prime $\ell \mid N$ which is non-split in K , and there is at least one such ℓ . Assume also that $\text{Gal}(\overline{\mathbf{Q}}/K) \rightarrow \text{Aut}_{\mathbf{Z}_p}(T)$ is surjective. Then*

$$(L_{p,1}^{\pm,\pm}(f/K)) = (\langle \mathbf{z}^{\pm}, \mathbf{z}^{\pm} \rangle_{K_{\infty}^{\text{ac}}}^{\text{cyc}})$$

as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Outline of the proof. In the ordinary case, Perrin-Riou's p -adic Gross–Zagier formula [PR87b] was first extended to the Λ -adic setting by Howard [How05] by a generalization of her methods. In the supersingular case, Kobayashi [Kob13] proved his p -adic Gross–Zagier formula also following the strategy laid out in [PR87b], namely, computing the Fourier coefficients of two p -adic modular forms—one related to the derivatives of the p -adic L -function and the other to the p -adic heights of Heegner points—and showing their equality. In contrast, the strategy that we follow here in our proof of Theorem B is via Iwasawa theory, based on the connections between Heegner points, Beilinson–Flach elements, and their explicit reciprocity laws.¹

More precisely, our proof of Theorem B may be divided into the following four steps.

Step 1: Explicit reciprocity law for the plus/minus Beilinson–Flach elements.

Recall that α and β denote the roots of $x^2 - a_p x + p$, each of which is a square root of $-p$, since $a_p = 0$ by assumption; in particular, both α and β have normalized p -adic valuation $1/2$. Let f_{α} (resp. f_{β}) be the p -stabilization of f with U_p -eigenvalue α (resp. β). In [LZ15], Loeffler and Zerbes have defined three-variable systems of cohomology classes $\mathcal{BF}_{\mathcal{F},\mathcal{G}}$ interpolating the Beilinson–Flach elements of [KLZ15] attached to the different specializations of two Coleman families \mathcal{F} and \mathcal{G} and their cyclotomic twists. Specializing their construction to \mathcal{F} equal the Coleman family passing through f_{α} and f_{β} , respectively, and \mathcal{G} equal to a certain Hida family of CM forms by K , we deduce two-variable classes

$$\mathcal{BF}_{\alpha}, \mathcal{BF}_{\beta} \in \mathbf{Q}_p[[\Gamma_K]] \otimes_{\Lambda[1/p]} H_{\text{Iw}}^1(K_{\infty}, T),$$

where $\Lambda := \mathbf{Z}_p[[\Gamma_K]]$. Moreover, from the main result of [LZ15] we have an explicit reciprocity law relating the image of these two-variable classes under a Perrin-Riou logarithm map to Loeffler's p -adic L -functions (0.2). Building upon these results, one can construct two bounded cohomology classes $\mathcal{BF}^{\pm} \in H_{\text{Iw}}^1(K_{\infty}, T)$ and four Λ -linear maps

$$\text{Col}^{\pm} : \frac{H_{\text{Iw}}^1(K_{\infty, \overline{\mathbf{p}}}, T)}{H_{\pm, \text{Iw}}^1(K_{\infty, \overline{\mathbf{p}}}, T)} \longrightarrow \mathbf{Z}_p[[\Gamma_K]] \quad \text{Log}^{\pm} : H_{\pm, \text{Iw}}^1(K_{\infty, \mathbf{p}}, T) \longrightarrow \mathbf{Z}_p[[\Gamma_K]],$$

such that

$$(0.7) \quad \text{Col}^{\circ}(\text{loc}_{\overline{\mathbf{p}}}(\mathcal{BF}^*)) = L_p^{*, \circ}(f/K), \quad \text{Log}^{\pm}(\text{loc}_{\mathbf{p}}(\mathcal{BF}^{\pm})) = \mathcal{L}_{\mathbf{p}}(f/K),$$

where $H_{\pm, \text{Iw}}^1(K_{\infty, \overline{\mathbf{p}}}, T) \subseteq H_{\text{Iw}}^1(K_{\infty, \overline{\mathbf{p}}}, T)$ is the local condition defining $\text{Sel}^{\pm, \pm}(K_{\infty}, T)$ at primes above $\overline{\mathbf{p}}$, and $\mathcal{L}_{\mathbf{p}}(f/K)$ is a certain p -adic Rankin–Selberg L -function constructed in [Wan15]. A more precise version of these results is explained in Section 3.1.

Step 2: Explicit reciprocity law and main conjecture for the plus/minus Heegner points.

In [How04b], Howard established one of the divisibilities predicted by Perrin-Riou's Heegner point main conjecture [PR87a]. More recently, the converse divisibility has been obtained by the second author in [Wan14]. As a key step towards the proof of Theorem B, in Section 4 we will formulate and prove the following analog of Perrin-Riou's Heegner point main conjecture for supersingular primes (with $a_p = 0$).

¹Inspired by the work of Agboola–Howard [AH06] in the ordinary CM case, a similar strategy was originally exploited by the first author in [Cas15] to give a new proof of Howard's Λ -adic Gross–Zagier formula [How05].

Theorem C. *Under the assumptions in Theorem B, both $\mathrm{Sel}^{\pm,\pm}(K_{\infty}^{\mathrm{ac}}, T)$ and the Pontryagin dual $\mathrm{Sel}_{p^{\infty}}^{\pm,\pm}(f/K_{\infty}^{\mathrm{ac}})^{\vee}$ have Λ^{ac} -rank one, and*

$$Ch_{\Lambda^{\mathrm{ac}}}(\mathrm{Sel}_{p^{\infty}}^{\pm,\pm}(f/K_{\infty}^{\mathrm{ac}})^{\vee}) = Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}^{\pm,\pm}(K_{\infty}^{\mathrm{ac}}, T)}{\Lambda^{\mathrm{ac}} \cdot \mathbf{z}^{\pm}}\right)^2$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, where the subscript ‘tors’ denotes the Λ^{ac} -torsion submodule.

This is obtained by combining the following three ingredients:

- (1) The divisibility \subseteq established in [Wan15] in the Iwasawa–Greenberg main conjecture

$$Ch_{\Lambda}(\mathrm{Sel}_{p^{\infty}}^{\mathrm{rel},\mathrm{str}}(f/K_{\infty})^{\vee}) \stackrel{?}{=} (\mathcal{L}_{\mathfrak{p}}(f/K)),$$

where $\mathrm{Sel}_{p^{\infty}}^{\mathrm{rel},\mathrm{str}}(f/K_{\infty})$ is defined by requiring local triviality (resp. no condition) at the places above $\bar{\mathfrak{p}}$ (resp. \mathfrak{p}).

- (2) An explicit reciprocity law for the plus/minus Heegner points

$$\mathrm{Log}_{\mathrm{ac}}^{\pm}(\mathrm{loc}_{\mathfrak{p}}(\mathbf{z}^{\pm})) = \mathcal{L}_{\mathfrak{p},\mathrm{ac}}(f/K) \cdot u,$$

where $\mathrm{Log}_{\mathrm{ac}}^{\pm}$ and $\mathcal{L}_{\mathfrak{p},\mathrm{ac}}(f/K)$ are the anticyclotomic projections of Log^{\pm} and $\mathcal{L}_{\mathfrak{p}}(f/K)$, respectively, and u is a unit in Λ^{ac} . This corresponds to Theorem 4.6 in the body of the paper, and it implies in particular that the classes $\mathrm{loc}_{\mathfrak{p}}(\mathbf{z}^{\pm})$ are not Λ^{ac} -torsion.

- (3) A Kolyvagin system argument involving the plus/minus Heegner classes \mathbf{z}^{\pm} , yielding the divisibility \supseteq in Theorem C.

Combined with the divisibility in [Wan15], the equality in Theorem C allows us to complete to proof the Iwasawa–Greenberg main conjecture for $\mathcal{L}_{\mathfrak{p}}(f/K)$.

Theorem D. *Under the assumptions in Theorem B, we have*

$$Ch_{\Lambda}(\mathrm{Sel}_{p^{\infty}}^{\mathrm{rel},\mathrm{str}}(f/K_{\infty})^{\vee}) = (\mathcal{L}_{\mathfrak{p}}(f/K))$$

as ideals in Λ^{ac} . Moreover, the equal-sign cases of Conjecture A hold.

This corresponds to Theorem 5.2 and Corollary 5.3 in the body of the paper.

Step 3: Rubin’s height formula.

Let $\mathcal{BF}_{\mathrm{ac}}^{\pm}$ denote the image of \mathcal{BF}^{\pm} under the natural map $H_{\mathrm{Iw}}^1(K_{\infty}, T) \rightarrow H_{\mathrm{Iw}}^1(K_{\infty}^{\mathrm{ac}}, T)$. As noted above, hypothesis (Heeg) forces the vanishing of the constant term of $L_p^{\pm,\pm}(f/K)$ in the expansion (0.4), or equivalently, of the anticyclotomic projection of $L_p^{\pm,\pm}(f/K)$. In light of the first explicit reciprocity law (0.7) (and the injectivity of Col^{\pm}), this implies that the classes $\mathcal{BF}_{\mathrm{ac}}^{\pm}$, which *a priori* lie in $\mathrm{Sel}^{\pm,\mathrm{rel}}(K_{\infty}^{\mathrm{ac}}, T)$, land in the smaller $\mathrm{Sel}^{\pm,\pm}(K_{\infty}^{\mathrm{ac}}, T)$.

Then, an analogue of Rubin’s height formula [Rub94] for the pairings (0.6) combined with a reformulation of Theorem D leads to a proof of the equality (see Theorem 5.4)

$$(L_{p,1}^{\pm,\pm}(f/K)) = R_{\mathrm{cyc}}^{\pm} \cdot Ch_{\Lambda^{\mathrm{ac}}}(\mathrm{Sel}_{p^{\infty}}^{\pm,\pm}(f/K_{\infty}^{\mathrm{ac}})^{\vee})$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, where R_{cyc}^{\pm} is the regulator of the Λ^{ac} -adic height pairing (0.6). Together with Theorem C, the proof of the Λ -adic Gross–Zagier formula in Theorem B follows immediately from this.

We end this Introduction by noting that related results have appeared in a recent preprint [BL16] by K. Büyükboduk and A. Lei. More precisely, [loc.cit., Thm. 4.5(v)] corresponds to the first part of Theorem C, and the $\mathcal{H}(\Gamma^{\mathrm{ac}})$ -adic Birch–Swinnerton-Dyer formula of [loc.cit., Thm. 5.31] should bear a close relationship with the two Λ^{ac} -adic Gross–Zagier formulae in our Theorem B. However our methods are markedly different: we use an Euler system argument for the plus/minus Heegner points \mathbf{z}^{\pm} , whereas the Euler system argument in [BL16] is applied to a variant of the plus/minus Beilinson–Flach classes \mathcal{BF}^{\pm} . As a consequence, the main results

in [BL16] are for the twists of f by a p -distinguished branch character, whereas we do not need to take any such twists here—in fact, and largely for simplicity, the case of untwisted f is the only case we consider here. Moreover, thanks to the explicit reciprocity laws of Theorem 3.1 (for \mathcal{BF}^\pm) and Theorem 4.6 (for \mathbf{z}^\pm), our results do not rely on any nonvanishing hypotheses. It would be interesting to see if our methods can be combined the constructions in [BL16] to yield a proof of some of the main conjectures formulated in *loc.cit.*, as well as an extension of our results to higher weights.

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1. p -ADIC L -FUNCTIONS

Throughout this section, we let $f = \sum_{n \geq 1} a_n(f)q^n \in S_2(\Gamma_0(N_f))$ be a newform, and K be an imaginary quadratic field of discriminant $-D_K < 0$ prime to N_f . Fix a prime $p \nmid 6N_f D_K$ and a choice of complex and p -adic embeddings $\mathbf{C} \xrightarrow{\iota_\infty} \overline{\mathbf{Q}} \xrightarrow{\iota_p} \mathbf{C}_p$. Since it will suffice for our application in this paper, for simplicity we shall assume that the number field generated by the Fourier coefficients $a_n(f)$ embeds into \mathbf{Q}_p .

1.1. p -adic Rankin–Selberg L -functions. Let Ξ_K denote the set of algebraic Hecke characters $\psi : K^\times \backslash \mathbb{A}_K^\times \rightarrow \mathbf{C}^\times$. We say that $\psi \in \Xi_K$ has infinity type $(\ell_1, \ell_2) \in \mathbf{Z}^2$ if

$$\psi_\infty(z) = z^{\ell_1} \bar{z}^{\ell_2},$$

where for each place v of K , we let $\psi_v : K_v^\times \rightarrow \mathbf{C}^\times$ be the v -component of ψ . The conductor of ψ is the largest ideal $\subseteq \mathcal{O}_K$ such that $\psi_{\mathfrak{q}}(u) = 1$ for all $u \in (1 + \mathcal{O}_{K, \mathfrak{q}})^\times \subseteq K_{\mathfrak{q}}^\times$. If ψ has conductor \mathfrak{c}_ψ and \mathfrak{a} is any fractional ideal of K prime to \mathfrak{c}_ψ , we write $\psi(\mathfrak{a})$ for $\psi(a)$, where a is an idele satisfying $a\hat{\mathcal{O}}_K \cap K = \mathfrak{a}$ and $a_{\mathfrak{q}} = 1$ for all \mathfrak{q} dividing \mathfrak{c}_ψ . As a function on fractional ideals, then ψ satisfies

$$\psi((\alpha)) = \alpha^{-\ell_1} \bar{\alpha}^{-\ell_2}$$

for all $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\psi}$.

We say that a Hecke character ψ of infinity type (ℓ_1, ℓ_2) is *critical* (for f) if $s = 1$ is a critical value in the sense of Deligne for

$$L(f/K, \psi, s) = L\left(\pi_f \times \pi_\psi, s + \frac{\ell_1 + \ell_2 - 1}{2}\right),$$

where $L(\pi_f \times \pi_\psi, s)$ is the L -function for the Rankin–Selberg convolution of the cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ associated to f and the theta series θ_ψ associated to ψ . Then the set of infinity types of critical characters can be written as the disjoint union

$$\Sigma = \Sigma^{(1)} \sqcup \Sigma^{(2)} \sqcup \Sigma^{(2')},$$

with $\Sigma^{(1)} = \{(0, 0)\}$, $\Sigma^{(2)} = \{(\ell_1, \ell_2) : \ell_1 \leq -1, \ell_2 \geq 1\}$, $\Sigma^{(2')} = \{(\ell_1, \ell_2) : \ell_2 \leq -1, \ell_1 \geq 1\}$.

The involution $\psi \mapsto \psi^\rho$ on Ξ_K , where ψ^ρ is obtained by composing ψ with the complex conjugation on \mathbb{A}_K^\times , has the effect on infinity types of interchanging the regions $\Sigma^{(2)}$ and $\Sigma^{(2')}$ (while leaving $\Sigma^{(1)}$ stable). Since the values $L(f/K, \psi, 1)$ and $L(f/K, \psi^\rho, 1)$ are the same, for the purposes of p -adic interpolation we may restrict our attention to the first two subsets in the above decomposition of Σ .

Definition 1.1. Let $\psi = \psi^\infty \psi_\infty \in \Xi_K$ be an algebraic Hecke character of infinity type (ℓ_1, ℓ_2) . The p -adic avatar $\hat{\psi} : K^\times \backslash \hat{K}^\times \rightarrow \mathbf{C}_p^\times$ of ψ defined by

$$\hat{\psi}(z) = \iota_p \iota_\infty^{-1}(\psi^\infty(z)) z_{\mathfrak{p}}^{\ell_1} \bar{z}_{\mathfrak{p}}^{\ell_2}.$$

For each ideal $\mathfrak{m} \subseteq \mathcal{O}_K$ let $H_{\mathfrak{m}} = \text{Gal}(K(\mathfrak{m})/K)$ denote the ray class group of K modulo \mathfrak{m} , and set $H_{p^\infty} = \varprojlim_r H_{p^r}$. Via the Artin reciprocity map, the correspondence $\psi \mapsto \hat{\psi}$ then establishes a bijection between the set of algebraic Hecke characters of K of conductor dividing p^∞ and the set of locally algebraic $\overline{\mathbf{Q}}_p$ -valued characters of H_{p^∞} .

Following the terminology of [Loe13, §2.3], for any $r, s \in \mathbf{R}_{\geq 0}$ we let $D^{(r,s)}(H_{p^\infty})$ be the space of \mathbf{Q}_p -valued distributions on H_{p^∞} of order (r, s) with respect to the quasi-factorization of H_{p^∞} induced by the ray class groups $H_{\mathfrak{p}^\infty}$ and $H_{\overline{\mathfrak{p}}^\infty}$ (see [loc.cit., Prop. 4]). For example, if $H_{p^\infty} \simeq H_{\mathfrak{p}^\infty} \times H_{\overline{\mathfrak{p}}^\infty}$, then $D^{(r,s)}(H_{p^\infty})$ may be identified with the dual of the completed tensor product $C^r(H_{\mathfrak{p}^\infty}) \hat{\otimes} C^s(H_{\overline{\mathfrak{p}}^\infty})$ equipped with the natural Banach space topology (see [Col10, §I.5]), where $C^r(H_{\mathfrak{p}^\infty})$ is the space of \mathbf{Q}_p -valued functions on $H_{\mathfrak{p}^\infty}$ of order r , and $C^s(H_{\overline{\mathfrak{p}}^\infty})$ is defined similarly.

For the statement of the next result, let M be a fixed positive integer divisible by $D_K N_f$ and having the same prime factors as $D_K N_f$.

Theorem 1.2. *Assume that $p = p\overline{p}$ splits in K , let α and β be the roots of $x^2 - a_p(f)x + p$, and set $r := v_p(\alpha)$ and $s := v_p(\beta)$.*

- (i) *There exists a p -adic L -function $L_p(f/K, \Sigma^{(2)}) \in \text{Frac}(\mathbf{Z}_p[[H_{p^\infty}]])$ such that for every character $\chi \in \Xi_K$ of trivial conductor and infinity type $(\ell_1, \ell_2) \in \Sigma^{(2)}$, we have*

$$L_p(f/K, \Sigma^{(2)})(\hat{\psi}) = \frac{\Gamma(\ell_2)\Gamma(\ell_2 + 1) \cdot \mathcal{E}(f, \psi)}{(1 - \psi^{1-\rho}(\mathfrak{p}))(1 - p^{-1}\psi^{1-\rho}(\mathfrak{p}))} \cdot \frac{L(f/K, \psi, 1)}{(2\pi)^{2\ell_2+1} \cdot \langle \theta_{\psi_{\ell_2}}, \theta_{\psi_{\ell_2}} \rangle_M},$$

where $\theta_{\psi_{\ell_2}}$ is the theta series of weight $\ell_2 - \ell_1 + 1 \geq 3$ associated to the Hecke character $\psi_{\ell_2} := \psi \mathbf{N}_K^{\ell_2}$ of infinity type $(\ell_1 - \ell_2, 0)$, and

$$\mathcal{E}(f, \psi) = (1 - p^{-1}\psi(\mathfrak{p})\alpha)(1 - p^{-1}\psi(\mathfrak{p})\beta)(1 - \psi^{-1}(\overline{\mathfrak{p}})\alpha^{-1})(1 - \psi^{-1}(\overline{\mathfrak{p}})\beta^{-1}).$$

- (ii) *If $r < 1$ and $s < 1$, then for each $\underline{\alpha} := (\alpha_{\mathfrak{p}}, \alpha_{\overline{\mathfrak{p}}}) \in \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)\}$ there exists an element $L_{p,\underline{\alpha}}(f/K, \Sigma^{(1)}) \in D^{(r,s)}(H_{p^\infty})$ such that for every finite order character $\psi \in \Xi_K$ of conductor $\mathfrak{c}_\psi \mid p^\infty$ we have*

$$L_{p,\underline{\alpha}}(f/K, \Sigma^{(1)})(\hat{\psi}) = \left(\prod_{\mathfrak{q} \mid p} \alpha_{\mathfrak{q}}^{-v_{\mathfrak{q}}(\mathfrak{c}_\psi)} \right) \cdot \frac{\mathcal{E}(\psi, f)}{\mathfrak{g}(\psi) \cdot N(\mathfrak{c}_\psi)^{1/2}} \cdot \frac{L(f/K, \psi, 1)}{(4\pi)^2 \cdot \langle f, f \rangle_M},$$

where

$$\mathcal{E}(\psi, f) = \prod_{\mathfrak{q} \mid p, \mathfrak{q} \nmid \mathfrak{c}_\psi} (1 - \alpha_{\mathfrak{q}}^{-1}\psi(\mathfrak{q}))(1 - \alpha_{\mathfrak{q}}^{-1}\psi^{-1}(\mathfrak{q})).$$

Proof. The first part is a reformulation of [LLZ15, Thm. 6.1.3(i)], and the second follows from [Loe13, Thm. 9] and [loc.cit., Prop. 7]. \square

Remark 1.3. The definition of $L_{p,\underline{\alpha}}(f/K, \Sigma^{(1)})$ in [Loe13] is done with a period Ω_Π attached to the base change to $\text{GL}_2(\mathbb{A}_K)$ for the cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ associated to f . However, it is easy to see that this differs from $\langle f, f \rangle_M$ by a nonzero factor in \mathbf{Q}^\times . The latter period is more convenient for our purposes, given its relation with the Rankin–Selberg p -adic L -functions constructed by Urban [Urb14] (see Theorem 3.1).

1.2. The two-variable plus/minus p -adic L -functions. Let $\Phi_n(X) = \sum_{i=0}^{p-1} X^{p^{n-1}i}$ be the p^n -th cyclotomic polynomial. Fix a topological generator $\gamma_v \in H_{v^\infty}$ for each prime $v \mid p$, and define the ‘half-logarithms’

$$\log_v^+ := \frac{1}{p} \prod_{m=1}^{\infty} \frac{\Phi_{2m}(\gamma_v)}{p}, \quad \log_v^- := \frac{1}{p} \prod_{m=1}^{\infty} \frac{\Phi_{2m-1}(\gamma_v)}{p}.$$

These are elements in $D^{1/2}(H_{v^\infty})$ which will be seen in $D^{(1/2, 1/2)}(H_{p^\infty})$ via pullback.

Theorem 1.4. *Assume that $a_p(f) = 0$. Then there exist four bounded \mathbf{Q}_p -valued measures on H_{p^∞} :*

$$L_p^{+,+}(f/K), L_p^{-,+}(f/K), L_p^{+,-}(f/K), L_p^{-,-}(f/K)$$

such that for every $\underline{\alpha} = (\alpha_p, \alpha_{\overline{p}})$ we have

$$\begin{aligned} L_{p,\underline{\alpha}}(f/K, \Sigma^{(1)}) &= L_p^{+,+}(f/K) \cdot \log_p^+ \log_{\overline{p}}^+ \\ &\quad + L_p^{-,+}(f/K) \cdot \log_p^- \log_{\overline{p}}^+ \cdot \alpha_p + L_p^{+,-}(f/K) \cdot \log_p^+ \log_{\overline{p}}^- \cdot \alpha_{\overline{p}} \\ &\quad + L_p^{-,-}(f/K) \cdot \log_p^- \log_{\overline{p}}^- \cdot \alpha_p \alpha_{\overline{p}}. \end{aligned}$$

Moreover, if ϕ is a finite order character of H_{p^∞} of conductor $\mathfrak{p}^{n_p} \overline{\mathfrak{p}}^{n_{\overline{p}}}$ with $n_p, n_{\overline{p}} > 0$, then $L_p^{*,\circ}(f/K)$ vanishes at ϕ unless $*$ or \circ is $(-1)^{n_p}$ and $(-1)^{n_{\overline{p}}}$.

Proof. This is shown in [Loe13, §5]. For our later use, we record the construction of the four $L_p^{*,\circ}(f/K)$ as an explicit linear combination of the four $L_{p,\underline{\alpha}} := L_{p,\underline{\alpha}}(f/K, \Sigma^{(1)})$. Fix a root α of the Hecke polynomial $x^2 - a_p(f)x + p = x^2 + p$, and let β be the other root. Then:

$$\begin{aligned} L_p^{+,+}(f/K) &= \frac{L_{p,(\alpha,\alpha)} + L_{p,(\beta,\alpha)} + L_{p,(\alpha,\beta)} + L_{p,(\beta,\beta)}}{4 \log_p^+ \log_{\overline{p}}^+}, \\ L_p^{-,+}(f/K) &= \frac{L_{p,(\alpha,\alpha)} - L_{p,(\beta,\alpha)} + L_{p,(\alpha,\beta)} - L_{p,(\beta,\beta)}}{4 \log_p^- \log_{\overline{p}}^+ \cdot \alpha}, \\ L_p^{+,-}(f/K) &= \frac{L_{p,(\alpha,\alpha)} + L_{p,(\beta,\alpha)} - L_{p,(\alpha,\beta)} - L_{p,(\beta,\beta)}}{4 \log_p^+ \log_{\overline{p}}^- \cdot \alpha}, \\ L_p^{-,-}(f/K) &= \frac{L_{p,(\alpha,\alpha)} - L_{p,(\beta,\alpha)} - L_{p,(\alpha,\beta)} + L_{p,(\beta,\beta)}}{4 \log_p^- \log_{\overline{p}}^- \cdot \alpha^2}. \end{aligned}$$

Using the relation $\beta = -\alpha$, it is immediate to check that the four identities (1.1) hold. \square

1.3. Anticyclotomic p -adic L -functions. Write

$$N_f = N^+ N^-$$

with N^+ (resp. N^-) equal to the product of the prime factors of N_f split (resp. inert) in K . We say that the pair (f, K) satisfies the *generalized Heegner hypothesis* if

(Heeg) N^- is the square-free product of an *even* number of primes.

Let K_∞/K be the \mathbf{Z}_p^2 -extension of K , and set $\Gamma_K = \text{Gal}(K_\infty/K)$. We may decompose

$$H_{p^\infty} \simeq \Delta \times \Gamma_K$$

with Δ a finite abelian group. The Galois group $\text{Gal}(K/\mathbf{Q})$ acts on Γ_K by conjugation. Let $\Gamma^{\text{cyc}} \subseteq \Gamma_K$ be the fixed part by this action, and set $\Gamma^{\text{ac}} := \Gamma_K / \Gamma^{\text{cyc}}$. Then $\Gamma^{\text{ac}} = \text{Gal}(K_\infty^{\text{ac}}/K)$ is the Galois group of the *anticyclotomic* \mathbf{Z}_p -extension of K , on which we have $\tau \sigma \tau^{-1} = \sigma^{-1}$ for the non-trivial element $\tau \in \text{Gal}(K/\mathbf{Q})$. Similarly, we say that a character ψ is *anticyclotomic* if $\psi(\tau \sigma \tau^{-1}) = \psi^{-1}$.

Let $L_{p,\underline{\alpha}}^{\text{ac}}(f/K)$ be the image of the p -adic L -function $L_{p,\underline{\alpha}}(f/K, \Sigma^{(1)})$ of Theorem 1.2 under the natural projection $D^{(r,s)}(H_{p^\infty}) \rightarrow D^{(r,s)}(\Gamma^{\text{ac}})$.

Theorem 1.5. *If the generalized Heegner hypothesis (Heeg) holds, then*

$$L_{p,(\alpha,\alpha)}^{\text{ac}}(f/K) \equiv 0 \quad \text{and} \quad L_{p,(\beta,\beta)}^{\text{ac}}(f/K) \equiv 0.$$

Proof. Via Rankin–Selberg convolution techniques, B.-D. Kim has constructed in [Kim14b] p -adic L -functions $\mathcal{L}_{p,(\alpha,\alpha)}(f/K)$ and $\mathcal{L}_{p,(\beta,\beta)}(f/K)$ which are easily seen to be nonzero constant multiples of Loeffler’s $L_{p,(\alpha,\alpha)}(f/K, \Sigma^{(1)})$ and $L_{p,(\beta,\beta)}(f/K, \Sigma^{(1)})$, respectively (see the remarks in [Loe13, p. 378]). Via the usual identifications $\mathbf{Q}_p[[H_{\mathfrak{p}^\infty}]] \simeq \mathbf{Q}_p[[X]]$ and $\mathbf{Q}_p[[H_{\overline{\mathfrak{p}}^\infty}]] \simeq \mathbf{Q}_p[[Y]]$ sending $\gamma_{\mathfrak{p}} \mapsto 1 + X$ and $\gamma_{\overline{\mathfrak{p}}} \mapsto 1 + Y$, we may view these p -adic L -functions as two-variable power series in the variables X and Y . Let ε_K denote the quadratic character associated with K by class field theory. The same argument as in [PR87b, Thm. 1.1] and [Dis15, §4.2], then shows that $\mathcal{L}_{p,(\alpha,\alpha)}(f/K)$ (and hence also $L_{p,(\alpha,\alpha)}(f/K, \Sigma^{(1)})$) satisfies the functional equation

$$(1.1) \quad \mathcal{L}_{p,(\alpha,\alpha)}(f/K) \left(\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1 \right) = \epsilon \mathcal{L}_{p,(\alpha,\alpha)}(f/K)(X, Y),$$

where $\epsilon = -\varepsilon_K(N)$, and similarly for $\mathcal{L}_{p,(\beta,\beta)}(f/K)$. Since the change of variables $(X, Y) \mapsto (\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1)$ corresponds to the transformation $\phi \mapsto \phi^{-\rho}$ on characters of H_{p^∞} , this shows that under the generalized Heegner hypothesis both $L_{p,(\alpha,\alpha)}(f/K, \Sigma^{(1)})$ and $L_{p,(\beta,\beta)}(f/K, \Sigma^{(1)})$ vanish at all anticyclotomic characters of H_{p^∞} (since $\epsilon = -1$), whence the result. \square

Throughout the following, we shall identify the space of bounded \mathbf{Q}_p -valued measures on a compact p -adic Lie group G with the Iwasawa algebra $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[G]]$. Viewing the measures $L_p^{\pm, \pm}(f/K)$ of Theorem 1.4 as elements in $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[H_{p^\infty}]]$, we thus denote by $L_{p,ac}^{\pm, \pm}(f/K)$ their images under the natural projection $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[H_{p^\infty}]] \rightarrow \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma^{ac}]]$.

Corollary 1.6. *Assume that $a_p(f) = 0$. If the generalized Heegner hypothesis (Heeg) holds, then*

$$L_{p,ac}^{\pm, \pm}(f/K) \equiv 0.$$

Proof. As in the proof of [Pol03, Thm. 5.13], the idea is to use the decomposition in Proposition 1.4 to deduce from the functional equation for $L_{p,\alpha}(f/K, \Sigma^{(1)})$ a similar one for $L_p^{\pm, \pm}(f/K)$ forcing the vanishing of $L_{p,ac}^{\pm, \pm}(f/K)$ under our generalized Heegner hypothesis.

Indeed, writing the functional equation (1.1) for $L_{p,(\alpha,\alpha)}(f/K, \Sigma^{(1)})$ in terms of the signed p -adic L -functions $L_p^{\pm, \pm} := L_p^{\pm, \pm}(f/K)$ we obtain

$$(1.2) \quad \begin{aligned} & \log_{\mathfrak{p}}^+ \log_{\overline{\mathfrak{p}}}^+ \cdot \left(L_p^{+,+}(X, Y) - \epsilon L_p^{+,+} \left(\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1 \right) \right) \\ & + \log_{\mathfrak{p}}^- \log_{\overline{\mathfrak{p}}}^- \cdot \left(L_p^{-,-}(X, Y) - \epsilon L_p^{-,-} \left(\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1 \right) \right) \cdot \alpha^2 \\ & = \log_{\mathfrak{p}}^+ \log_{\overline{\mathfrak{p}}}^- \cdot \left(-L_p^{+,-}(X, Y) + \epsilon L_p^{+,-} \left(\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1 \right) \right) \cdot \alpha \\ & + \log_{\mathfrak{p}}^- \log_{\overline{\mathfrak{p}}}^+ \cdot \left(-L_p^{-,+}(X, Y) + \epsilon L_p^{-,+} \left(\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1 \right) \right) \cdot \alpha, \end{aligned}$$

where $\epsilon = -\varepsilon_K(N)$. Since $v_p(\alpha) = 1/2$, the nonzero coefficients in the left-hand side of this equality have coefficients with p -adic valuations in \mathbf{Z} , whereas the nonzero coefficients in the right-hand side have p -adic valuations in $\frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$. This forces both sides to be identically zero, and so we obtain

$$L_p^{+,+}(f/K) \left(\frac{1}{1+Y} - 1, \frac{1}{1+X} - 1 \right) = \epsilon L_p^{+,+}(f/K)(X, Y),$$

and similarly for $L_p^{-,-}(f/K)$. In particular, it follows that $L_{p,ac}^{+,+}(f/K) \equiv 0$ and $L_{p,ac}^{-,-}(f/K) \equiv 0$ under the generalized Heegner hypothesis. On the other hand, since the p -adic L -functions $L_p^{+,-}(f/K)$ and $L_p^{-,+}(f/K)$ vanish at all characters ϕ of H_{p^∞} whose conductors at \mathfrak{p} and $\overline{\mathfrak{p}}$ have the same parity, we also have $L_{p,ac}^{+,-}(f/K) \equiv 0$ and $L_{p,ac}^{-,+}(f/K) \equiv 0$. \square

A key role in this paper will be played by the following anticyclotomic p -adic L -function, whose construction relies on an explicit form of Waldspurger's special value formula. Let R_0 denote the completion of the ring of integers of the maximal unramified extension of \mathbf{Q}_p .

Theorem 1.7. *Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and that (Heeg) holds. Then there exists a p -adic L -function $\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K) \in R_0[[\Gamma^{\text{ac}}]]$ such that if $\hat{\psi} : \Gamma^{\text{ac}} \rightarrow \mathbf{C}_p^\times$ has trivial conductor and infinity type $(-\ell, \ell)$ with $\ell \geq 1$, then*

$$\left(\frac{\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)(\hat{\psi})}{\Omega_p^{2\ell}} \right)^2 = \Gamma(\ell)\Gamma(\ell+1) \cdot (1 - p^{-1}\psi(\mathfrak{p})\alpha)^2 (1 - p^{-1}\psi(\mathfrak{p})\beta)^2 \cdot \frac{L(f/K, \psi, 1)}{\pi^{2\ell+1} \cdot \Omega_K^{4\ell}},$$

where $\Omega_p \in R_0^\times$ and $\Omega_K \in \mathbf{C}^\times$ are CM periods attached to K .

Proof. This follows from the results in [CH15, §3.3]. (See the proof of Theorem 4.6 below for the precise relation between the construction in *loc.cit.* and the above $\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)$.) \square

The next nonvanishing result for the anticyclotomic p -adic L -function in Theorem 1.7 will play an important role in our arguments. Let $\rho_f : \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{Aut}_{\mathbf{Q}_p}(V_f) \simeq \text{GL}_2(\mathbf{Q}_p)$ be the Galois representation attached to f , and let $\bar{\rho}_f$ denote its associated semi-simple mod p representation.

Theorem 1.8. *Assume in addition that*

- $\bar{\rho}_f|_{\text{Gal}(\bar{\mathbf{Q}}/K)}$ is absolutely irreducible,
- $\bar{\rho}_f$ is ramified at every prime $\ell|N^-$.

Then $\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)$ is not identically zero, and it has trivial μ -invariant.

Proof. The nonvanishing of $\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)$ follows from [CH15, Thm. 3.7], where it is deduced from [Hsi14, Thm. C]. The vanishing of $\mu(\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K))$ similarly follows from [Hsi14, Thm. B] (or alternatively, from [Bur14, Thm. B] in the cases where the number of prime factors in N^- is positive, noting that by the discussion in [Pra06, p. 912] our last assumption guarantees that the term $\alpha(f, f_B)$ in [Bur14, Thm. 5.6] is a p -adic unit). \square

Letting $L_{p,\text{ac}}(f/K)$ be the image of the p -adic L -function $L_p(f/K, \Sigma^{(2)})$ of Theorem 1.2 under the natural projection $\text{Frac}(\mathbf{Z}_p[[H_{p^\infty}]] \rightarrow \text{Frac}(\mathbf{Z}_p[[\Gamma^{\text{ac}}]])$, we note that $L_{p,\text{ac}}(f/K)$ and the square of the p -adic L -function $\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)$ are defined by the interpolation of the *same* L -values. However, the archimedean periods used in their construction are different, and so these p -adic L -functions need not be equal (even up to units in the Iwasawa algebra). In fact, as shown in Theorem 1.10 below, the ratio between these different periods is interpolated by an anticyclotomic projection of a Katz p -adic L -function.²

Before we state the defining property of the Katz p -adic L -function, recall that the Hecke L -function of $\psi \in \Xi_K$ is defined by (the analytic continuation of) the Euler product

$$L(\psi, s) = \prod_{\mathfrak{l}} \left(1 - \frac{\psi(\mathfrak{l})}{N(\mathfrak{l})^s} \right)^{-1},$$

where \mathfrak{l} runs over all prime ideals of K , with the convention that $\psi(\mathfrak{l}) = 0$ for $\mathfrak{l}|\mathfrak{c}_\psi$. The set of infinity types of $\psi \in \Xi_K$ for which $s = 0$ is a critical value of $L(\psi, s)$ can be written as the disjoint union $\Sigma_K \sqcup \Sigma'_K$, where $\Sigma_K = \{(\ell_1, \ell_2) : 0 < \ell_1 \leq \ell_2\}$ and $\Sigma'_K = \{(\ell_1, \ell_2) : 0 < \ell_2 \leq \ell_1\}$.

²This phenomenon appears to have been first observed by Hida–Tilouine [HT93, §8] in a slightly different context; see also [DLR15, §3.2].

Theorem 1.9. *Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K . Then there is a p -adic L -function $\mathcal{L}_{\mathfrak{p}}^{\text{Katz}}(K) \in R_0[[H_{p^\infty}]]$ such that if $\psi \in \Xi_K$ has trivial conductor and infinity type $(\ell_1, \ell_2) \in \Sigma_K$, then*

$$\frac{\mathcal{L}_{\mathfrak{p}}^{\text{Katz}}(K)(\hat{\psi})}{\Omega_p^{\ell_2 - \ell_1}} = \left(\frac{\sqrt{D_K}}{2\pi} \right)^{\ell_1} \cdot \Gamma(\ell_2) \cdot (1 - \psi(\mathfrak{p}))(1 - p^{-1}\psi^{-1}(\bar{\mathfrak{p}})) \cdot \frac{L(\psi, 0)}{\Omega_K^{\ell_2 - \ell_1}},$$

where Ω_p and Ω_K are as in Theorem 1.7.

Proof. See [Kat78, §5.3.0], or [dS87, Thm. II.4.14]. \square

Denote by $\mathcal{L}_{\mathfrak{p}, \text{ac}}^{\text{Katz}}(K)$ the image of $\mathcal{L}_{\mathfrak{p}}^{\text{Katz}}(K)$ under the projection $R_0[[H_{p^\infty}]] \rightarrow R_0[[\Gamma^{\text{ac}}]]$.

Theorem 1.10. *Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and that (Heeg) holds. Then*

$$L_{p, \text{ac}}(f/K)(\hat{\psi}) = \frac{w_K}{h_K} \cdot \frac{\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)^2(\hat{\psi})}{\mathcal{L}_{\mathfrak{p}, \text{ac}}^{\text{Katz}}(K)(\hat{\psi}^{\rho-1})}$$

up to a unit in $\mathbf{Z}_p[[\Gamma^{\text{ac}}]]^\times$, where $w_K = |\mathcal{O}_K^\times|$ and h_K is the class number of K .

Proof. This is [Cas15, Thm. 1.7], whose proof does not make use of the underlying (in *loc.cit.*) ordinarity hypothesis on f . See also [JSW15, §5.3]. \square

1.4. Another p -adic Rankin–Selberg L -function. Recall the decomposition $H_{p^\infty} \simeq \Delta \times \Gamma_K$, set $\Lambda := \mathbf{Z}_p[[\Gamma_K]]$ and $\Lambda_{R_0} := R_0[[\Gamma_K]]$, and denote also by

$$L_p(f/K, \Sigma^{(2)}) \in \text{Frac}(\Lambda) \quad \text{and} \quad \mathcal{L}_{\mathfrak{p}}^{\text{Katz}}(K) \in \Lambda_{R_0}$$

the natural projections of the p -adic L -functions $L_p(f/K, \Sigma^{(2)})$ and $\mathcal{L}_{\mathfrak{p}}^{\text{Katz}}(K)$ of Theorem 1.2 and Theorem 1.9, respectively.

Theorem 1.11. *Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K . There exists a p -adic L -function*

$$\mathcal{L}_{\mathfrak{p}}(f/K) \in \Lambda_{R_0}$$

such that if $\hat{\psi} : \Gamma \rightarrow \mathbf{C}^\times$ has trivial conductor and infinity type $(\ell_1, \ell_2) \in \Sigma^{(2)}$, then

$$\mathcal{L}_{\mathfrak{p}}(f/K)(\hat{\psi}) = \frac{\Gamma(\ell_2)\Gamma(\ell_2+1)}{\pi^{2\ell_2+1}} \cdot \mathcal{E}(f, \psi) \cdot \frac{\Omega_p^{2(\ell_2-\ell_1)}}{\Omega_K^{2(\ell_2-\ell_1)}} \cdot L(f/K, \psi, 1),$$

where $\mathcal{E}(f, \psi) = (1 - p^{-1}\psi(\mathfrak{p})\alpha)(1 - p^{-1}\psi(\mathfrak{p})\beta)(1 - \psi^{-1}(\bar{\mathfrak{p}})\alpha^{-1})(1 - \psi^{-1}(\bar{\mathfrak{p}})\beta^{-1})$, and Ω_K and Ω_p are as in Theorem 1.7. Moreover, $\mathcal{L}_{\mathfrak{p}}(f/K)$ differs from the product

$$(1.3) \quad \widetilde{\mathcal{L}}_{\mathfrak{p}}(f/K)(\hat{\psi}) := L_p(f/K, \Sigma^{(2)})(\hat{\psi}) \cdot \frac{h_K}{w_K} \cdot \mathcal{L}_{\mathfrak{p}, \text{ac}}^{\text{Katz}}(K)(\hat{\psi}^{\rho-1})$$

by a unit in Λ^\times , and it is not identically zero.

Proof. The construction of $\mathcal{L}_{\mathfrak{p}}(f/K)$ is given in [Wan15, §4.6]. On the other hand, the fact that the product (1.3) has the claimed interpolation property follows by a straightforward adaptation of the calculations in [Cas15, Thm. 1.7]. Finally, the fact that $\mathcal{L}_{\mathfrak{p}}(f/K)$ is nonzero follows from the fact that for some of the characters ψ in the range of p -adic interpolation, the Euler product defining $L(f, \psi, s)$ converges at $s = 1$. \square

Corollary 1.12. *Assume that $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K and (Heeg) holds, and let $\mathcal{L}_{\mathfrak{p}, \text{ac}}(f/K)$ be the image of the p -adic L -function $\mathcal{L}_{\mathfrak{p}}(f/K)$ of Theorem 1.11 under the anticyclotomic projection $\Lambda_{R_0} \rightarrow \Lambda_{R_0}^{\text{ac}}$. Then*

$$\mathcal{L}_{\mathfrak{p}, \text{ac}}(f/K) = \mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)^2$$

up to a unit in $(\Lambda^{\text{ac}})^\times$, where $\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)$ is as in Theorem 1.7.

Proof. This follows from a direct comparison of their interpolation properties. \square

2. SELMER GROUPS

2.1. Local conditions at p . In this section, we develop some local results for studying the anticyclotomic Iwasawa theory for elliptic curves at supersingular primes. Throughout, we let E/\mathbf{Q} be an elliptic curve of conductor N , $p \geq 5$ be a prime of good supersingular reduction for E , and K/\mathbf{Q} be an imaginary quadratic field of discriminant prime to N and such that

$$p = \mathfrak{p}\bar{\mathfrak{p}} \quad \text{splits in } K.$$

Let $\Phi_m(X) = \sum_{i=0}^{p^m-1} X^{p^{m-1}i}$ be the p^m -th cyclotomic polynomial, and define

$$\tilde{\omega}_n^+(X) := \prod_{\substack{1 \leq m \leq n \\ m \text{ even}}} \Phi_m(X+1), \quad \tilde{\omega}_n^-(X) := \prod_{\substack{1 \leq m \leq n \\ m \text{ odd}}} \Phi_m(X+1),$$

and set $\omega_n^\pm(X) = X\tilde{\omega}_n^\pm(X)$.

It is easy to see that every prime $v|p$ is finitely decomposed in K_∞^{ac}/K , say as the product $v_1 v_2 \cdots v_{p^t}$; then v is also decomposed into p^t distinct primes in K_∞/K . Let Γ_1 (resp. Γ_1^{ac}) be the decomposition group of v_1 in Γ (resp. Γ^{ac}). Let H_K be the Hilbert class field of K , set $K_0^{\text{ac}} := K_\infty^{\text{ac}} \cap H_K$, and let K_m^{ac} be the subfield of K_∞^{ac} with $[K_m^{\text{ac}} : K_0^{\text{ac}}] = p^m$. Let a be the inertial degree of K_0^{ac}/K at any $v|p$.

We denote by \mathbf{Q}_p^{nr} and $\mathbf{Q}_{p,\infty}$ the unramified and the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q}_p , respectively, and let $\mathbf{Q}_{p,\infty}^{\text{nr}}$ denote their composition. For $v|p$, we identify $K_v \simeq \mathbf{Q}_p$. Let u_v and γ_v be topological generators of $U_v := \text{Gal}(\mathbf{Q}_{p,\infty}^{\text{nr}}/\mathbf{Q}_{p,\infty})$ and $\Gamma_v := \text{Gal}(\mathbf{Q}_{p,\infty}^{\text{nr}}/\mathbf{Q}_p^{\text{nr}})$; these are chosen so that u_v is the arithmetic Frobenius and $-p^a u_v + \gamma_v$ is a topological generator of $\text{Gal}(K_{\infty,v}/K_{\infty,v}^{\text{ac}})$. Let $X_v = \gamma_v - 1$ and $Y_v = u_v - 1$. Finally, let $\gamma^{\text{ac}} \in \Gamma^{\text{ac}}$ be a topological generator, so that $\mathbf{Z}_p[[\Gamma^{\text{ac}}]] \simeq \mathbf{Z}_p[[T]]$ setting $T = \gamma^{\text{ac}} - 1$.

Following [Kob03] (see also [Kim14a, §2.1]), for any unramified extension k of \mathbf{Q}_p we define the subgroups $E^\pm(k(\mu_{p^{n+1}}))$ of $E(k(\mu_{p^{n+1}}))$ by

$$E^+(k(\mu_{p^{n+1}})) = \left\{ x \in E(k(\mu_{p^{n+1}})) \mid \text{tr}_{k(\mu_{p^{\ell+1}})}^{k(\mu_{p^{n+1}})}(x) \in E(k(\mu_{p^{\ell+1}})) \text{ for } 0 \leq \ell < n, \text{ even } \ell \right\},$$

$$E^-(k(\mu_{p^{n+1}})) = \left\{ x \in E(k(\mu_{p^{n+1}})) \mid \text{tr}_{k(\mu_{p^{\ell+1}})}^{k(\mu_{p^{n+1}})}(x) \in E(k(\mu_{p^{\ell+1}})) \text{ for } -1 \leq \ell < n, \text{ odd } \ell \right\}.$$

Letting \hat{E} denote the formal group associated to the minimal model of E over \mathbf{Z}_p , we may similarly define the subgroups $\hat{E}^\pm(\mathfrak{m}_{k(\mu_{p^{n+1}})})$ of $\hat{E}(\mathfrak{m}_{k(\mu_{p^{n+1}})})$, and the assumption that $a_p = 0$ easily implies that

$$E^\pm(k(\mu_{p^{n+1}})) \otimes \mathbf{Q}_p/\mathbf{Z}_p = \hat{E}^\pm(\mathfrak{m}_{k(\mu_{p^{n+1}})}) \otimes \mathbf{Q}_p/\mathbf{Z}_p.$$

Fix a compatible system $\{\zeta_{p^n}\}_{n \geq 0}$ of primitive p^n -th roots of unity ζ_{p^n} (i.e., $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for $n > 0$ and $\zeta_p \neq 1$). Let φ be the Frobenius on k/\mathbf{Q}_p , and for any $f \in k[X]$ set

$$\log_f(X) = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(2n)}(X)}{p^n},$$

where $f^{(2n)}(X) = f^{\varphi^{2n-1}} \circ \cdots \circ f^\varphi \circ f(X)$. As in [Kim07, §3.2], for any unit $z \in \mathcal{O}_k^\times$ one can construct a point $\tilde{c}_{n,z} \in \hat{E}(\mathfrak{m}_{k(\mu_{p^n})})$ such that

$$(2.1) \quad \log_{\hat{E}}(\tilde{c}_{n,z}) = \left[\sum_{i=1}^{\infty} (-1)^{i-1} z^{\varphi^{-(n+2i)}} \cdot p^i \right] + \log_{f_z^{\varphi^{-n}}}(z^{\varphi^{-n}} \cdot (\zeta_{p^n} - 1)),$$

with $f_z(X) := (X + z)^p - z^p$. Since $\hat{E}(\mathfrak{m}_{k(\mu_{p^n})})$ is torsion-free (see [Kob03, Prop. 8.7] and [Kim07, Prop. 3.1]), the formal group logarithm $\log_{\hat{E}}$ is injective, and hence the point $\tilde{c}_{n,z}$ is uniquely defined by (2.1).

Let k_n be the subextension of $k(\mu_{p^{n+1}})$ of degree p^n over k , and let $\mathfrak{m}_{k,n}$ be the maximal ideal of its valuation ring. Denote by $\hat{E}^\pm(\mathfrak{m}_{k,n})$ the image of $\hat{E}^\pm(\mathfrak{m}_{k(\mu_{p^{n+1}})})$ under the trace map $\mathrm{tr}_{k_n}^{k(\mu_{p^{n+1}})}$, define $E^\pm(k_n)$ similarly, and set

$$(2.2) \quad c_{n,z} := \mathrm{tr}_{k_n}^{k(\mu_{p^{n+1}})}(\tilde{c}_{n+1,z}) \in \hat{E}(\mathfrak{m}_{k,n}).$$

Let k^m be the unramified extension of \mathbf{Q}_p of degree p^m , write $k_{n,m}$ and $\mathfrak{m}_{n,m}$ for the above k_n and $\mathfrak{m}_{k,n}$ with $k = k^m$, and set

$$\begin{aligned} \Lambda_{n,m} &:= \mathbf{Z}_p[\mathrm{Gal}(k_{n,m}/\mathbf{Q}_p)], & \Lambda_{n,m}^\pm &:= \mathbf{Z}_p[[\Gamma_1]]/(\omega_n^\pm(X), (1+Y)^{p^m} - 1) \\ & & & \simeq \tilde{\omega}_n^\mp(X) \Lambda_{n,m}, \end{aligned}$$

where the last isomorphism follows from the relation $(1+X)^{p^n} - 1 = X\tilde{\omega}_n^+(X)\tilde{\omega}_n^-(X)$.

Lemma 2.1. *There is a sequence of points $c_{n,m} \in \hat{E}(\mathfrak{m}_{n,m})$ satisfying the compatibilities:*

$$\mathrm{tr}_{k_{n,m-1}}^{k_{n,m}}(c_{n,m}) = c_{n,m-1}, \quad \mathrm{tr}_{k_{n-1,m}}^{k_{n,m}}(c_{n,m}) = -c_{n-2,m}.$$

Moreover, for even (resp. odd) n , $c_{n,m}$ generates $\hat{E}^+(\mathfrak{m}_{n,m})$ (resp. $\hat{E}^-(\mathfrak{m}_{n,m})$) over $\Lambda_{n,m}$.

Proof. This first part is [Wan15, Lemma 6.2]. For our later use, we recall the construction of $c_{n,m}$. By the normal basis theorem, we may fix an element $d = \{d_m\}_m \in \varprojlim_m \mathcal{O}_{k^m}^\times$ generating $\varprojlim_m \mathcal{O}_{k^m}^\times$ as a $\mathbf{Z}_p[[U]]$ -module. Writing $d_m = \sum_j a_{m,j} \zeta_j$, with ζ_j roots of unity and $a_{m,j} \in \mathbf{Z}_p$, one then defines

$$(2.3) \quad c_{n,m} := \sum_j a_{m,j} c_{n,\zeta_j},$$

where c_{n,ζ_j} is as in (2.2). The proof of the equalities in the lemma then follows from an explicit calculations of the images under $\log_{\hat{E}}$ of both sides using (2.1). The second claim then follows from [Wan15, Lemma 6.4]. \square

Definition 2.2. We define $H_\pm^1(k_{n,m}, T) \subseteq H^1(k_{n,m}, T)$ to be the orthogonal complement of $E^\pm(k_{n,m}) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ under the local Tate pairing

$$(\ , \)_{n,m} : H^1(k_{n,m}, T) \times H^1(k_{n,m}, E[p^\infty]) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p,$$

where we view $E^\pm(k_{n,m}) \otimes \mathbf{Q}_p/\mathbf{Z}_p$ as embedded in $H^1(k_{n,m}, E[p^\infty])$ by the Kummer map.

2.2. The plus/minus Coleman maps. In this section, we briefly recall Kobayashi's construction of the plus/minus Coleman maps for the cyclotomic \mathbf{Z}_p -extension of \mathbf{Q}_p , as adapted by Kim [Kim14a] to finite unramified extensions of \mathbf{Q}_p . We keep the notations introduced in Section 2.1.

Define the maps $P_{c_{n,m}} : H^1(k_{n,m}, T) \rightarrow \mathbf{Z}_p[\mathrm{Gal}(k_{n,m}/\mathbf{Q}_p)]$ by

$$P_{c_{n,m}}(z) = \sum_{\sigma \in \mathrm{Gal}(k_{n,m}/\mathbf{Q}_p)} (c_{n,m}^\sigma, z)_{n,m},$$

and set $P_{c_{n,m}}^\pm := (-1)^{[\frac{n+1}{2}]} P_{c_{n,m}}^\pm$, where

$$c_{n,m}^+ = \begin{cases} c_{n,m} & \text{if } n \text{ is even,} \\ c_{n-1,m} & \text{if } n \text{ is odd,} \end{cases} \quad c_{n,m}^- = \begin{cases} c_{n-1,m} & \text{if } n \text{ is even,} \\ c_{n,m} & \text{if } n \text{ is odd.} \end{cases}$$

By Lemma 2.1, the maps $P_{c_{n,m}}^\pm$ factor through the quotient by $H_\pm^1(k_{n,m}, T)$ and they satisfy natural compatibilities for varying n and m . Moreover, as shown in [Kim14a, Thms. 2.7-8] (see

also [Kob03, §8.5]), there are unique maps $\text{Col}_{n,m}^\pm$ making the following diagram commutative:

$$\begin{array}{ccc} H^1(k_{n,m}, T) & \xrightarrow{\text{Col}_{n,m}^\pm} & \Lambda_{n,m}^\pm \\ \downarrow & & \downarrow \cdot \tilde{\omega}_n^\mp \\ H^1(k_{n,m}, T)/H_\pm^1(k_{n,m}, T) & \xrightarrow{P_{c_{n,m}}^\pm} & \Lambda_{n,m}. \end{array}$$

The maps $\text{Col}_{n,m}^\pm$ are isomorphisms and passing to the limit they define Λ -linear isomorphisms

$$(2.4) \quad \text{Col}^\pm : \varprojlim_{n,m} \frac{H^1(k_{n,m}, T)}{H_\pm^1(k_{n,m}, T)} \xrightarrow{\sim} \varprojlim_{n,m} \Lambda_{n,m} \simeq \mathbf{Z}_p[[\Gamma_1]].$$

2.3. The plus/minus logarithm maps. We now define local big logarithm maps $\text{Log}_{\text{ac}}^\pm$ on $H_\pm^1(K_v, \mathbf{T}^{\text{ac}})$, where

$$(2.5) \quad \mathbf{T}^{\text{ac}} := T \otimes \mathbf{Z}_p[[\Gamma^{\text{ac}}]](\Psi^{-1})$$

for the canonical character $\Psi : \Gamma^{\text{ac}} \hookrightarrow \mathbf{Z}_p[[\Gamma^{\text{ac}}]]^\times$. As it will be clear to the reader, these maps are the restriction to the ‘anticyclotomic line’ of the two-variable plus/minus logarithm maps Log^\pm introduced in [Wan15, §6.1]. We still keep the notations from Section 2.1.

Via the natural inclusion

$$E(k_{n,m}) \otimes \mathbf{Q}_p/\mathbf{Z}_p = (E(k_{n,m}) \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\perp \subseteq (E^\pm(k_{n,m}) \otimes \mathbf{Q}_p/\mathbf{Z}_p)^\perp = H_\pm^1(k_{n,m}, T),$$

we may view the points $c_{n,m}$ in the $\Lambda_{n,m}$ -module $H_\pm^1(k_{n,m}, T)$. Then, by [Wan15, Lemma 6.9] one can choose norm-compatible classes $b_{n,m}^\pm \in H_\pm^1(k_{n,m}, T)$ with the property that

$$\tilde{\omega}_n^{-\epsilon}(X)b_{n,m}^\epsilon = (-1)^{[\frac{n+1}{2}]}c_{n,m},$$

where $\epsilon = (-1)^n$, and such that $\varprojlim_{n,m} b_{n,m}^\pm$ generates $\varprojlim_{m,n} H_\pm^1(k_{n,m}, T)$ as a free $\mathbf{Z}_p[[\Gamma_1]]$ -module of rank one. Noting that $K_{m,v}^{\text{ac}} \subseteq k_{m,m+a}$, we define

$$\begin{aligned} E^+(K_{m,v}^{\text{ac}}) &= \left\{ x \in E(K_{m,v}^{\text{ac}}) \mid \text{tr}_{K_{\ell+1,v}^{\text{ac}}}^{K_{m,v}^{\text{ac}}}(x) \in E(K_{\ell,v}^{\text{ac}}) \text{ for } 0 \leq \ell < m, \text{ even } \ell \right\}, \\ E^-(K_{m,v}^{\text{ac}}) &= \left\{ x \in E(K_{m,v}^{\text{ac}}) \mid \text{tr}_{K_{\ell+1,v}^{\text{ac}}}^{K_{m,v}^{\text{ac}}}(x) \in E(K_{\ell,v}^{\text{ac}}) \text{ for } -1 \leq \ell < m, \text{ odd } \ell \right\}, \end{aligned}$$

and we easily see that

$$E^\pm(K_{m,v}^{\text{ac}}) \otimes \mathbf{Q}_p/\mathbf{Z}_p = (E^\pm(k_{m,m+a}) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \cap H^1(K_{m,v}^{\text{ac}}, E[p^\infty]).$$

Let $H_\pm^1(K_{m,v}^{\text{ac}}, T)$ be the image of $H_\pm^1(k_{m,m+a}, T)$ under corestriction from $k_{m,m+a}$ to K_m^{ac} . Set $\mathbf{T}_1^{\text{ac}} = T \otimes \mathbf{Z}_p[[\Gamma_1]](\Psi^{-1})$, where $\Psi : \Gamma_1^{\text{ac}} \hookrightarrow \mathbf{Z}_p[[\Gamma_1^{\text{ac}}]]^\times$ is the canonical character, and we let G_{K_v} act diagonally on the tensor product \mathbf{T}_1^{ac} . Then $H_\pm^1(K_v, \mathbf{T}_1^{\text{ac}}) \simeq \varprojlim_m H^1(K_{m,v}^{\text{ac}}, T)$ by Shapiro’s lemma, and the elements

$$a_m^\pm := \text{tr}_{K_{m,v}^{\text{ac}}}^{k_{m,m+a}}(b_{m,m+a}^\pm)$$

are norm-compatible, with $a^\pm := \varprojlim_m a_m^\pm$ generating $H_\pm^1(K_v, \mathbf{T}_1^{\text{ac}})$ as a free $\mathbf{Z}_p[[\Gamma_1^{\text{ac}}]]$ -module.

Recall that v_1, v_2, \dots, v_{p^t} denote the primes over a place $v|p$ in the extension K_∞/K . Since every prime above p is totally ramified in $K_\infty/K_\infty^{\text{ac}}$, we will still denote by v_1, v_2, \dots, v_{p^t} , the primes above v in K_∞^{ac} . Let $\gamma_1 = \text{id}, \gamma_2, \dots, \gamma_{p^t} \in \Gamma^{\text{ac}}$ be such that $\gamma_i v_1 = v_i$. Then we have the direct sum decompositions

$$(2.6) \quad \mathbf{Z}_p[[\Gamma^{\text{ac}}]] = \bigoplus_{i=1}^{p^t} \gamma_i \mathbf{Z}_p[[\Gamma_1^{\text{ac}}]], \quad H_\pm^1(K_v, \mathbf{T}^{\text{ac}}) = \bigoplus_{i=1}^{p^t} \gamma_i H_\pm^1(K_v, \mathbf{T}_1^{\text{ac}}).$$

Definition 2.3. For every prime $v|p$ in K , define the map

$$\mathrm{Log}_{\mathrm{ac}}^{\pm} : H_{\pm}^1(K_v, \mathbf{T}_1^{\mathrm{ac}}) \longrightarrow \mathbf{Z}_p[[\Gamma_1^{\mathrm{ac}}]]$$

by the relation

$$x = \mathrm{Log}_{\mathrm{ac}}^{\pm}(x) \cdot a^{\pm}$$

for all $x \in H_{\pm}^1(K_v, \mathbf{T}_1^{\mathrm{ac}})$. Also, let $\mathrm{Log}_{\mathrm{ac}}^{\pm} : H_{\pm}^1(K_v, \mathbf{T}^{\mathrm{ac}}) \rightarrow \mathbf{Z}_p[[\Gamma^{\mathrm{ac}}]]$ be the natural extension of the above map using (2.6). This does not depend on the choice of γ_i in the decompositions.

The following result establishes the interpolation property satisfied by the map $\mathrm{Log}_{\mathrm{ac}}^+$ (the result for $\mathrm{Log}_{\mathrm{ac}}^-$ is entirely similar).

Lemma 2.4. *Let $\phi : \Gamma^{\mathrm{ac}} \rightarrow \mathbf{C}_p^{\times}$ be a finite order character of conductor p^n , with $n > 0$ even. If $x = \varprojlim_n x_n \in H_+^1(K_v^{\mathrm{ac}}, \mathbf{T}^{\mathrm{ac}})$, then the following formulas hold:*

$$\begin{aligned} \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) \log_{\hat{E}}(x_n^{\tau}) \cdot \tilde{\omega}_n^{-}(\phi) &= \phi^{-1}(\mathrm{Log}_{\mathrm{ac}}^+(x)) \cdot (-1)^{n/2} \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) \log_{\hat{E}}(c_{n,n+a}^{\tau}), \\ \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) \log_{\hat{E}}(c_{n,n+a}^{\tau}) &= \frac{\mathfrak{g}(\phi)}{\phi(p^n)} \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) d_{n+a}^{\tau}. \end{aligned}$$

Proof. The first equality follows directly from the definitions. On the other hand, an immediate calculation using (2.1) and (2.3) (cf. [Wan15, Lemma 6.2]) reveals that

$$\log_{\hat{E}}(c_{n,n+a}) = \sum_i (-1)^{i-1} d_{n+a}^{\varphi^{-(n+2i)}} \cdot p^i + \sum_{0 \leq 2k < n} (-1)^k d_{n+a}^{\varphi^{2k-n}} \cdot \frac{\zeta_{p^{n-2k}} - 1}{p^k}.$$

Thus we find that

$$\begin{aligned} \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) \log_{\hat{E}}(c_{n,n+a}^{\tau}) &= \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) d_{n+a}^{\varphi^{-n}\tau} \cdot (\zeta_{p^n}^{\tau} - 1) \\ &= \mathfrak{g}(\phi) \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) d_{n+a}^{\varphi^{-n}\tau} = \frac{\mathfrak{g}(\phi)}{\phi(p^n)} \sum_{\tau \in \Gamma^{\mathrm{ac}}/p^n \Gamma^{\mathrm{ac}}} \phi(\tau) d_{n+a}^{\tau}. \end{aligned}$$

□

Remark 2.5. With an eye towards the eventual generalization of the results of this paper to modular forms of higher weight, it would be interesting to reformulate the construction of the maps $\mathrm{Log}_{\mathrm{ac}}^{\pm}$ in terms of Perrin-Riou's big logarithm maps [PR94], [LZ14], similarly as done in [Lei11] for Kobayashi's plus/minus Coleman maps.

2.4. The two-variable plus/minus Selmer groups. Let T denote the p -adic Tate module of E , viewed as a G_K -representation, and set $V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $A = V/T \simeq E[p^{\infty}]$. Let Σ be a finite set of places of K containing the places above p and ∞ and the primes where V ramifies, and for any finite extension F of K contained in K_{∞} , let $\mathfrak{G}_{F,\Sigma}$ denote the Galois group of the maximal extension of F unramified outside the places above Σ .

Following [BK90], for every place v of F we define

$$H_f^1(F_v, V) := \begin{cases} \ker [H^1(F_v, V) \rightarrow H^1(F_v^{\mathrm{nr}}, V)] & \text{if } v \nmid p; \\ \ker [H^1(F_v, V) \rightarrow H^1(F_v, V \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}})] & \text{if } v \mid p, \end{cases}$$

where $\mathbf{B}_{\mathrm{cris}}$ is Fontaine's ring of crystalline periods.

Definition 2.6. The *Bloch–Kato Selmer group* of V over F is defined by

$$\mathrm{Sel}(F, V) = \ker \left[H^1(\mathfrak{G}_{F,\Sigma}, V) \rightarrow \prod_v \frac{H^1(F_v, V)}{H_f^1(F_v, V)} \right],$$

where v runs over all places of F .

Replacing the local conditions $H_f^1(F_v, V) \subseteq H^1(K_v, V)$ by their natural images in $H^1(F_v, A)$ (resp. preimages in $H^1(F_v, T)$), we similarly define $\text{Sel}(F, A) \subseteq H^1(\mathfrak{G}_{F, \Sigma}, A)$ (resp. $\text{Sel}(F, T) \subseteq H^1(\mathfrak{G}_{F, \Sigma}, T)$).

We will also have use for the following modified Selmer groups, obtained from the above by changing the local condition at the places above p . For $v|p$ and $\mathcal{L}_v \in \{\text{rel}, \pm, \text{str}\}$, set

$$H_{\mathcal{L}_v}^1(F_v, V) := \begin{cases} H^1(F_v, V) & \text{if } \mathcal{L}_v = \text{rel}; \\ H_{\pm}^1(F_v, V) & \text{if } \mathcal{L}_v = \pm; \\ \{0\} & \text{if } \mathcal{L}_v = \text{str}, \end{cases}$$

and for $\mathcal{L} = \{\mathcal{L}_v\}_{v|p}$, define

$$\text{Sel}_{\mathcal{L}}(F, V) := \ker \left[H^1(\mathfrak{G}_{F, \Sigma}, V) \longrightarrow \prod_{v \nmid p} \frac{H^1(F_v, V)}{H_f^1(F_v, V)} \times \prod_{v|p} \frac{H^1(F_v, V)}{H_{\mathcal{L}_v}^1(F_v, V)} \right].$$

As above, we can also define $\text{Sel}_{\mathcal{L}}(F, T)$ and $\text{Sel}_{\mathcal{L}}(F, A)$ using the tautological exact sequence $0 \rightarrow T \rightarrow V \rightarrow A \rightarrow 0$.

Recall that $\Gamma_K = \text{Gal}(K_{\infty}/K)$ denotes the Galois group of the \mathbf{Z}_p^2 -extension of K , and define the $\Lambda = \mathbf{Z}_p[[\Gamma_K]]$ -modules

$$\mathbf{T} := T \otimes_{\mathbf{Z}_p} \Lambda(\Psi^{-1}), \quad \mathbf{A} := \mathbf{T} \otimes_{\Lambda} \text{Hom}_{\mathbf{Z}_p}(\Lambda, \mathbf{Q}_p/\mathbf{Z}_p),$$

where $\Psi : \Gamma_K \hookrightarrow \Lambda^{\times}$ is the map sending $\gamma \in \Gamma_K$ to the corresponding group-like element in Λ^{\times} , $\Lambda(\Psi^{-1})$ denotes the free Λ -module of rank 1 where Γ_K acts via Ψ^{-1} , and the Γ_K -action on the tensor product defining \mathbf{T} is the diagonal one. The Selmer groups $\text{Sel}_{\mathcal{L}}(K, \mathbf{T})$ and $\text{Sel}_{\mathcal{L}}(K, \mathbf{A})$ may be defined similarly as before, and by Shapiro's lemma we then have that

$$\text{Sel}_{\mathcal{L}}(K, \mathbf{T}) \simeq \varprojlim_{K \subseteq F \subseteq K_{\infty}} \text{Sel}_{\mathcal{L}}(F, T)$$

where the limit is with respect to the corestriction maps as F runs over the finite extensions of K contained in K_{∞} (see [SU14, Prop. 3.4], for example). Also, define

$$X_{\mathcal{L}}(K, \mathbf{A}) := \text{Hom}_{\mathbf{Z}_p}(\text{Sel}_{\mathcal{L}}(K, \mathbf{A}), \mathbf{Q}_p/\mathbf{Z}_p).$$

For the ease of notation, set

$$\Lambda^{\text{ac}} := \mathbf{Z}_p[[\Gamma^{\text{ac}}]],$$

where we recall that $\Gamma^{\text{ac}} = \text{Gal}(K_{\infty}^{\text{ac}}/K)$ is the Galois group of the anticyclotomic \mathbf{Z}_p -extension of K . Then the Selmer groups $\text{Sel}_{\mathcal{L}}(K, \mathbf{T}^{\text{ac}})$ and $X_{\mathcal{L}}(K, \mathbf{A}^{\text{ac}})$ are defined by replacing Γ_K and Λ with Γ^{ac} and Λ^{ac} in the above definitions.

Denote by $\iota : \Lambda^{\text{ac}} \rightarrow \Lambda^{\text{ac}}$ the involution given by inversion on group-like elements, and for any Λ^{ac} -module M , let M^{ι} denote the underlying module M with the Λ^{ac} -module structure given by $\Lambda^{\text{ac}} \xrightarrow{\iota} \Lambda^{\text{ac}} \rightarrow \text{Aut}(M)$. Finally, let M_{tors} denote the Λ^{ac} -torsion submodule of M .

Lemma 2.7. *Assume that $E[p]|_{G_K}$ is absolutely irreducible. Then:*

- (1) $\text{rank}_{\Lambda^{\text{ac}}} \text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}}) = \text{rank}_{\Lambda^{\text{ac}}} X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})$.
- (2) $\text{rank}_{\Lambda^{\text{ac}}} X^{\pm, \text{rel}}(K, \mathbf{A}^{\text{ac}}) = 1 + \text{rank}_{\Lambda^{\text{ac}}} X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})$.
- (3) $Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{rel}}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}) = Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})_{\text{tors}})$ up to powers of $p\Lambda^{\text{ac}}$.

Proof. The first statement follows easily from [How04b, Prop. 2.2.8] (cf. [Wan14, Lemma 3.5]), so we just need to prove (2) and (3), for which we essentially follow the arguments in [AH06, §1.2]. By [MR04, Lemma 3.5.3, Thm. 4.1.13], for every continuous character $\psi : \Gamma^{\text{ac}} \rightarrow L^{\times}$ with values in some finite extension L/\mathbf{Q}_p with ring of integers \mathfrak{O}_L , there is a noncanonical isomorphism

$$(2.7) \quad H_{\pm, \text{rel}}^1(K, \mathbf{A}^{\text{ac}}(\psi))[p^i] \simeq (L/\mathfrak{O}_L)^r[p^i] \oplus H_{\pm, \text{str}}^1(K, \mathbf{A}^{\text{ac}}(\psi^{-1}))[p^i]$$

for all $i \geq 0$. Here $H_{\pm, \text{rel}}^1(K, \mathbf{A}^{\text{ac}}(\psi)) \subseteq \text{Sel}^{\pm, \text{rel}}(K, \mathbf{A}^{\text{ac}}(\psi))$ is the generalized Selmer group consisting of classes whose restriction at \mathfrak{p} lies in $H^1(K_{\mathfrak{p}}, \mathbf{A}^{\text{ac}}(\psi))_{\text{div}}$, while $H_{\pm, \text{str}}^1(K, \mathbf{A}^{\text{ac}}(\psi^{-1}))$ is the same as $\text{Sel}^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}}(\psi^{-1}))$, and r is the *core rank* (see [MR04, Def. 4.1.11]) of the Selmer conditions defining $H_{\pm, \text{rel}}^1(K, \mathbf{A}^{\text{ac}}(\psi))$, which by [DDT94, Thm. 2.18] it is given by the quantity

$$(2.8) \quad \text{corank}_{\mathfrak{O}_L} H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{A}^{\text{ac}}(\psi)) + \text{corank}_{\mathfrak{O}_L} H^1(K_{\overline{\mathfrak{p}}}, \mathbf{A}^{\text{ac}}(\psi)) - \text{corank}_{\mathfrak{O}_L} H^0(K_w, \mathbf{A}^{\text{ac}}(\psi)),$$

where w denotes the infinite place of K . By the local Euler characteristic formula, the first two terms in (2.8) are equal to 2 and 1 respectively, while the third one clearly equals 2. Thus $r = 1$ in (2.7) and letting $i \rightarrow \infty$ we conclude that

$$(2.9) \quad H_{\pm, \text{rel}}^1(K, \mathbf{A}^{\text{ac}}(\psi)) \simeq (L/\mathfrak{O}_L) \oplus H_{\pm, \text{str}}^1(K, \mathbf{A}^{\text{ac}}(\psi^{-1})).$$

Now, it is easy to show that the natural restriction maps

$$\begin{aligned} H_{\pm, \text{rel}}^1(K, \mathbf{A}^{\text{ac}}(\psi)) &\longrightarrow \text{Sel}^{\pm, \text{rel}}(K, \mathbf{A}^{\text{ac}})(\psi)^{\Gamma^{\text{ac}}} \\ H_{\pm, \text{str}}^1(K, \mathbf{A}^{\text{ac}}(\psi^{-1})) &\longrightarrow \text{Sel}^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})(\psi^{-1})^{\Gamma^{\text{ac}}} \end{aligned}$$

are injective with finite bounded cokernel as ψ varies (cf. [AH06, Lemma 1.2.4]), and since

$$\text{Sel}^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})(\psi^{-1})^{\Gamma^{\text{ac}}} \simeq \text{Sel}^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})(\psi)^{\Gamma^{\text{ac}}}$$

by the action of complex conjugation, we see that statements (2) and (3) follow from (2.9) by the same argument as in [AH06, Lemma 1.2.6]. \square

3. BEILINSON–FLACH ELEMENTS

3.1. The plus/minus Beilinson–Flach elements. In this section, building on the work of Loeffler–Zerbes [LZ14] on Beilinson–Flach elements in Coleman families, we show the existence of certain plus/minus classes which map to the plus/minus p -adic L -functions of §1.2 under the plus/minus Coleman maps. Similar results were first obtained in [Wan15, §7.3], but our proofs here are slightly different.

We maintain the set-up introduced in Section 2.1, and let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ be the normalized newform associated with E . In particular, $a_p = 0$.

Recall the $\mathbf{Z}_p[[\Gamma_1]]$ -linear maps Col^{\pm} introduced in (2.4), and extend them (using the analog of the decomposition (2.6) with Γ in place of Γ^{ac}) to Λ -linear isomorphisms

$$\text{Col}^{\pm} : \frac{H^1(K_v, \mathbf{T})}{H_{\pm}^1(K_v, \mathbf{T})} \longrightarrow \Lambda$$

for each prime v above p .

Theorem 3.1. *There exist elements $\mathcal{BF}^{\pm} \in \text{Sel}^{\pm, \text{rel}}(K, \mathbf{T})$ such that*

$$\text{Col}^{\pm}(\text{res}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\pm})) = u \cdot h^{\pm} \cdot L_p^{\pm, \pm}(f/K), \quad \text{Log}^{\pm}(\text{res}_{\mathfrak{p}}(\mathcal{BF}^{\pm})) = h^{\pm} \cdot \mathcal{L}_{\mathfrak{p}}(f/K),$$

for some nonzero $u, h^{\pm} \in \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. This is shown in [Wan15, §7.3], where it is deduced from the work of Loeffler–Zerbes [LZ15] on Beilinson–Flach elements in Coleman families and their explicit reciprocity laws. The elements h^{\pm} arise from the construction of the integral classes \mathcal{BF}^{\pm} , while u is the ratio between two different periods attached to a certain Hida family of CM forms, whose integrality (up to powers of p) is shown in [Wan15, §8.1] building on the work of Rubin [Rub91] and Hida–Tilouine [HT94]. Finally, we also note that the maps Log^{\pm} in the statement are constructed in [Wan15, Def. 6.10], and the map $\text{Log}_{\text{ac}}^{\pm}$ in Definition 2.3 of this paper are their restriction to the anticyclotomic line. \square

Remark 3.2. By the construction in [Wan15, §7.3], the element h^\pm divides the half-logarithm \log_p^\pm . Since the latter does not vanish identically along the anticyclotomic line, the same is true for h^\pm .

3.2. Two-variable main conjectures. As we shall show below, Theorem 3.1 can be used to essentially establish the equivalence of the following three different variants of the Iwasawa main conjecture for elliptic curves at supersingular primes.

- (1) The main conjecture ‘without p -adic zeta functions’ for the plus/minus Selmer groups.
- (2) The Iwasawa–Greenberg main conjecture for $\mathcal{L}_p(f/K)$.
- (3) The equal-sign cases of Kim’s two-variable main conjectures [Kim14a].

The p -adic L -function $\mathcal{L}_p(f/K)$ was constructed in Theorem 1.11 as an element in Λ_{R_0} , but (as we show in §4.1, for example) the corresponding principal ideal in Λ_{R_0} can be generated by an element $\mathcal{L}_p(f/K) \in \Lambda$. Thus the aforementioned relation takes the following form.

Theorem 3.3. *The following two statements are equivalent:*

- (1) $X^{\text{rel},\text{str}}(K, \mathbf{A})$ is Λ -torsion, and

$$Ch_\Lambda(X^{\text{rel},\text{str}}(K, \mathbf{A})) = (\mathcal{L}_p(f/K))$$

as ideals in Λ .

- (2) $X^{\pm,\text{str}}(K, \mathbf{A})$ is Λ -torsion, $\text{Sel}^{\pm,\text{rel}}(K, \mathbf{T})$ has Λ -rank 1, and

$$Ch_\Lambda(X^{\pm,\text{str}}(K, \mathbf{A})) \cdot \mathcal{H}^\pm = Ch_\Lambda\left(\frac{\text{Sel}^{\pm,\text{rel}}(K, \mathbf{T})}{\Lambda \cdot \mathcal{BF}^\pm}\right),$$

where $\mathcal{H}^\pm \subseteq \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the ideal generated by the element h^\pm in Theorem 3.1.

Moreover, $X^{\pm,\pm}(K, \mathbf{A})$ is Λ -torsion, and if any of the above two statements hold, then

$$Ch_\Lambda(X^{\pm,\pm}(K, \mathbf{A})) \subseteq (L_p^{\pm,\pm}(f/K)),$$

with equality if and only if the element u in Theorem 3.1 is invertible.

Proof. This is essentially shown in [Wan15, §8.1]. For the convenience of the reader, we briefly recall the argument. Poitou–Tate global duality gives rise to the exact sequences

$$(3.1) \quad 0 \longrightarrow \text{Sel}^{\pm,\text{rel}}(K, \mathbf{T}) \longrightarrow H_\pm^1(K_p, \mathbf{T}) \longrightarrow X^{\text{rel},\text{str}}(K, \mathbf{A}) \longrightarrow X^{\pm,\text{str}}(K, \mathbf{A}) \longrightarrow 0,$$

$$(3.2) \quad 0 \longrightarrow \text{Sel}^{\pm,\text{rel}}(K, \mathbf{T}) \longrightarrow \frac{H^1(K_{\overline{\mathbf{p}}}, \mathbf{T})}{H_\pm^1(K_{\overline{\mathbf{p}}}, \mathbf{T})} \longrightarrow X^{\pm,\pm}(K, \mathbf{A}) \longrightarrow X^{\pm,\text{str}}(K, \mathbf{A}) \longrightarrow 0,$$

where exactness on the leftmost terms relies on the nonvanishing of $L_p^{\pm,\pm}(f/K)$ and $\mathcal{L}_p(f/K)$. The control theorem of [Wan15, Prop. 8.7] combined with [Kob03, Thm. 1.2] implies that $X^{\pm,\pm}(K, \mathbf{A})$ is Λ -torsion. By (3.2), it follows that $X^{\pm,\text{str}}(K, \mathbf{A})$ is Λ -torsion and $\text{Sel}^{\pm,\text{rel}}(K, \mathbf{T})$ has Λ -rank one, and by (3.2), that $X^{\text{rel},\text{str}}(K, \mathbf{A})$ is Λ -torsion. Finally, [Wan15, Cor. 7.9] and Theorem 3.1 yield the following exact sequences from the above:

$$0 \longrightarrow \frac{\text{Sel}^{\pm,\text{rel}}(K, \mathbf{T})}{\Lambda \cdot \mathcal{BF}^\pm} \longrightarrow \frac{\Lambda}{\mathcal{H}^\pm \cdot (\mathcal{L}_p(f/K))} \longrightarrow X^{\text{rel},\text{str}}(K, \mathbf{A}) \longrightarrow X^{\pm,\text{str}}(K, \mathbf{A}) \longrightarrow 0,$$

$$0 \longrightarrow \frac{\text{Sel}^{\pm,\text{rel}}(K, \mathbf{T})}{\Lambda \cdot \mathcal{BF}^\pm} \longrightarrow \frac{\Lambda}{\mathcal{H}^\pm \cdot \mathcal{U} \cdot (L_p^{\pm,\pm}(f/K))} \longrightarrow X^{\pm,\pm}(K, \mathbf{A}) \longrightarrow X^{\pm,\text{str}}(K, \mathbf{A}) \longrightarrow 0,$$

where $\mathcal{U} := (u) \subseteq \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the ideal generated by the element u in Theorem 3.1. By the multiplicativity of characteristic ideals along exact sequences, the result follows. \square

Corollary 3.4. *If any of the equivalent statements in Theorem 3.3 hold, then*

$$Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})) \cdot \mathcal{H}_{\text{ac}}^{\pm} = Ch_{\Lambda^{\text{ac}}} \left(\frac{\text{Sel}^{\pm, \text{rel}}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathcal{BF}_{\text{ac}}^{\pm}} \right),$$

where $\mathcal{BF}_{\text{ac}}^{\pm}$ is the image of \mathcal{BF}^{\pm} under the projection $\text{Sel}^{\pm, \text{rel}}(K, \mathbf{T}) \rightarrow \text{Sel}^{\pm, \text{rel}}(K, \mathbf{T}^{\text{ac}})$ and $\mathcal{H}_{\text{ac}}^{\pm} \subseteq \Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the anticyclotomic projection of \mathcal{H}^{\pm} .

Proof. This follows from descending the equality in Theorem 3.3(1) from K_{∞} to K_{∞}^{ac} . Let $\gamma^{\text{cyc}} \in \Gamma^{\text{cyc}}$ be a topological generator, and let I^{cyc} be the principal ideal $(\gamma^{\text{cyc}} - 1)\Lambda \subseteq \Lambda$. Then by [SU14, Prop. 3.9] (with the roles of the cyclotomic and anticyclotomic \mathbf{Z}_p -extensions reversed) we have

$$X^{\pm, \text{str}}(K, \mathbf{A})/I^{\text{cyc}} X^{\pm, \text{str}}(K, \mathbf{A}) \simeq X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}}),$$

and by [Rub91, Lemma 6.2(ii)] it follows that

$$(3.3) \quad Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})) = Ch_{\Lambda}(X^{\pm, \text{str}}(K, \mathbf{A})) \cdot \mathfrak{D},$$

where $\mathfrak{D} := Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{str}}(K, \mathbf{A})[I^{\text{cyc}}])$. On the other hand, set

$$Z(K_{\infty}) := \text{Sel}^{\pm, \text{rel}}(K, \mathbf{T})/(\mathcal{BF}^{\pm}), \quad Z(K_{\infty}^{\text{ac}}) := \text{Sel}^{\pm, \text{rel}}(K, \mathbf{T}^{\text{ac}})/(\mathcal{BF}_{\text{ac}}^{\pm}).$$

Using the fact that I^{cyc} is principal, a straightforward application of the snake lemma yields the exactness of

$$(3.4) \quad \text{Sel}^{\pm, \text{rel}}(K, \mathbf{T})[I^{\text{cyc}}] \longrightarrow Z(K_{\infty})[I^{\text{cyc}}] \longrightarrow (\mathcal{BF}^{\pm})/I^{\text{cyc}}(\mathcal{BF}^{\pm}).$$

Arguing as in the proof of [AH06, Prop. 2.4.15] we see that the natural Λ^{ac} -module map

$$Z(K_{\infty})/I^{\text{cyc}} Z(K_{\infty}) \longrightarrow Z(K_{\infty}^{\text{ac}})$$

is injective with cokernel having characteristic ideal \mathfrak{D} , and hence

$$(3.5) \quad Ch_{\Lambda^{\text{ac}}}(Z(K_{\infty}^{\text{ac}})) = Ch_{\Lambda^{\text{ac}}}(Z(K_{\infty})/I^{\text{cyc}} Z(K_{\infty})) \cdot \mathfrak{D}.$$

Now the leftmost term in (3.4) vanishes by Theorem 4.10 and the rightmost one is clearly torsion-free, and hence $Z(K_{\infty})[I^{\text{cyc}}]$ is torsion-free. Since [Rub91, Lemma 6.2(i)] and equality (3.5) imply that $Z(K_{\infty})[I^{\text{cyc}}]$ is also a torsion Λ^{ac} -module (using the nonvanishing of the terms in that equality), we conclude that $Z(K_{\infty})[I^{\text{cyc}}] = 0$, and by [Rub91, Lemma 6.2(ii)] it follows that

$$(3.6) \quad Ch_{\Lambda}(Z(K_{\infty})) \cdot \Lambda^{\text{ac}} = Ch_{\Lambda^{\text{ac}}}(Z(K_{\infty})/I^{\text{cyc}} Z(K_{\infty})).$$

Combined with (3.3), we thus arrive at

$$\begin{aligned} Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})) &= Ch_{\Lambda}(X^{\pm, \text{str}}(K, \mathbf{A})) \cdot \mathfrak{D} \\ &= \mathcal{H}^{\pm} \cdot Ch_{\Lambda}(Z(K_{\infty})) \cdot \mathfrak{D} \\ &= \mathcal{H}_{\text{ac}}^{\pm} \cdot Ch_{\Lambda^{\text{ac}}}(Z(K_{\infty}^{\text{ac}})), \end{aligned}$$

using (3.5) and (3.6) for the last equality. This completes the proof. \square

3.3. Rubin's height formula. We keep the notations introduced in §1.3, and still denote by $L_p^{\pm, \pm}(f/K)$ the image of the p -adic L -function $L_p^{\pm, \pm}(f/K)$ of Proposition 1.4 under the projection

$$\mathbf{Z}_p[[H_{p^{\infty}}]] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

Let $\gamma^{\text{cyc}} \in \Gamma^{\text{cyc}}$ be a topological generator, and using the identification $\Lambda \simeq \Lambda^{\text{ac}}[[\Gamma^{\text{cyc}}]]$ expand

$$(3.7) \quad L_p^{\pm, \pm}(f/K) = L_{p,0}^{\pm, \pm}(f/K) + L_{p,1}^{\pm, \pm}(f/K)(\gamma^{\text{cyc}} - 1) + \dots$$

as a power series in $\gamma^{\text{cyc}} - 1$ with coefficients in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Thus $L_{p,0}^{\pm, \pm}(f/K)$ is nothing but the anticyclotomic projection $L_{p, \text{ac}}^{\pm, \pm}(f/K)$, which vanishes by Theorem 1.5. By Theorem 3.1

(and the injectivity of Col^\pm), this implies that the image $\mathcal{BF}_{\text{ac}}^\pm$ of the classes \mathcal{BF}^\pm under the canonical projection

$$H^1(K, \mathbf{T}) \longrightarrow H^1(K, \mathbf{T}^{\text{ac}})$$

lands in $\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}}) \subseteq \text{Sel}^{\pm, \text{rel}}(K, \mathbf{T}^{\text{ac}})$. We similarly define $h_0^\pm, u_0 \in \Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ to be the constant term in the expansion of the elements h^\pm and u in Theorem 3.1, so that $\mathcal{H}_{\text{ac}}^\pm = (h_0^\pm)$ in the notations of Corollary 3.4.

In the following, we let $L_n = K_n^{\text{ac}} K_\infty^{\text{cyc}}$, and think of $\mathcal{BF}^\pm \in H^1(K, \mathbf{T}) \simeq H_{\text{Iw}}^1(K_\infty, T)$ as a compatible system of classes $\mathcal{BF}_{\text{cyc}, n}^\pm \in H_{\text{Iw}}^1(L_n, T)$.

Lemma 3.5. *For every $n \geq 0$ there is a unique element*

$$\beta_n^\pm \in \frac{H_{\text{Iw}}^1(L_{n, \bar{\mathfrak{p}}}, T)}{H_{\text{Iw}, \pm}^1(L_{n, \bar{\mathfrak{p}}}, T)} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

such that

$$(\gamma^{\text{cyc}} - 1)\beta_n^\pm = \text{loc}_{\bar{\mathfrak{p}}}(\mathcal{BF}_{\text{cyc}, n}^\pm).$$

Letting $\beta_n^\pm(\mathbb{1})$ be the image of β_n^\pm in $H^1(K_{n, \bar{\mathfrak{p}}}^{\text{ac}}, T)/H_\pm^1(K_{n, \bar{\mathfrak{p}}}^{\text{ac}}, T)[1/p]$, the elements $\beta_n^\pm(\mathbb{1})$ define an element $\beta_\infty^\pm(\mathbb{1}) \in H^1(K_{\bar{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})/H_\pm^1(K_{\bar{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})[1/p]$, and the maps Col^\pm yield identifications

$$\frac{H^1(K_{\bar{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})}{H_\pm^1(K_{\bar{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \simeq \Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

sending $\beta_\infty^\pm(\mathbb{1})$ to the product $u_0 \cdot h_0^\pm \cdot L_{p, 1}^{\pm, \pm}(f/K)$.

Proof. Since $u_0 \cdot h_0^\pm \cdot L_{p, 1}^{\pm, \pm}(f/K)$ is clearly the coefficient of $\gamma^{\text{cyc}} - 1$ in the cyclotomic expansion of $u \cdot h^\pm \cdot L_p^{\pm, \pm}(f/K)$, the result follows from the definition of $\beta_\infty^\pm(\mathbb{1})$ and Theorem 3.1. \square

Let $\mathcal{I} \subseteq \mathbf{Z}_p[[\Gamma^{\text{cyc}}]]$ be the augmentation ideal, and set $\mathcal{J} = \mathcal{I}/\mathcal{I}^2$.

Theorem 3.6. *For every $n \geq 0$ there is a canonical (up to sign) p -adic height pairing*

$$\langle \cdot, \cdot \rangle_{K_n^{\text{ac}}}^{\text{cyc}} : \text{Sel}^{\pm, \pm}(K_n^{\text{ac}}, T)[1/p] \times \text{Sel}^{\pm, \pm}(K_n^{\text{ac}}, T)[1/p] \longrightarrow p^{-k} \mathbf{Z}_p \otimes_{\mathbf{Z}_p} \mathcal{J}$$

for some $k \in \mathbf{Z}_{\geq 0}$ independent of n , such that for every $b \in \text{Sel}^{\pm, \pm}(K_n^{\text{ac}}, T)[1/p]$, we have

$$(3.8) \quad \langle \mathcal{BF}_{\text{cyc}, n}^\pm(\mathbb{1}), b \rangle_{K_n^{\text{ac}}}^{\text{cyc}} = (\beta_n^\pm(\mathbb{1}), \text{loc}_{\mathfrak{p}}(b))_n \otimes (\gamma^{\text{cyc}} - 1),$$

where $(\cdot, \cdot)_n$ is the \mathbf{Q}_p -linear extension of the local Tate pairing

$$\frac{H^1(K_{n, \bar{\mathfrak{p}}}^{\text{ac}}, T)}{H_\pm^1(K_{n, \bar{\mathfrak{p}}}^{\text{ac}}, T)} \times H_\pm^1(K_{n, \mathfrak{p}}^{\text{ac}}, T) \longrightarrow \mathbf{Z}_p.$$

Proof. Since by [Kim07, Prop. 4.11] the plus/minus local conditions are their own orthogonal complement under the local Tate pairing, the construction of the cyclotomic p -adic height pairing in our setting can be deduced from [How04a, Thm. 1.11]. The p -adic height formula (3.8) then follows from [How04a, Thm. 2.5(c)], extending earlier work by Rubin [Rub94]. \square

4. HEEGNER POINTS

Let E/\mathbf{Q} be an elliptic curve with square-free conductor N , let $f = \sum_{n=1}^\infty a_n q^n \in S_2(\Gamma_0(N))$ be the associated newform, and assume that $p \geq 5$ is a prime of good supersingular reduction for E . In particular, we have $a_p = 0$. Let K be an imaginary quadratic field in which $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits. Throughout this section, we shall assume that the pair (f, K) satisfies the generalized Heegner hypothesis (Heeg) introduced in §1.3.

4.1. The plus/minus Heegner classes. In this section, we construct the plus/minus Heegner classes over the anticyclotomic \mathbf{Z}_p -extension K_∞^{ac}/K . As first observed by Darmon–Iovita [DI08], in the supersingular setting the classes directly constructed from Heegner points are not compatible under corestriction, and one is led to ‘divide’ these classes by certain products of p -power cyclotomic polynomials. Thus our construction resembles that in *loc.cit.*, but with a key difference: we work with the \mathbf{Z}_p -free module $T = T_p(E)$ rather than its quotients modulo p^n . As a consequence, the freeness result exploited in the construction (cf. [DI08, Prop. 3.20] and Lemma 4.3 below) becomes much simpler to prove in our case.

Recall that we let $K[m]$ denote the ring class field of K of conductor m .

Proposition 4.1. *For every $m \geq 1$ coprime to N , there are Heegner points $P[m] \in E(K[m])$ such that*

$$\text{tr}_{K[m^{p^{k+1}}]}^{K[m^{p^{k+2}}]}(P[m^{p^{k+2}}]) = a_p P[m^{p^{k+1}}] - P[m^{p^k}]$$

for all $k \geq 0$.

Proof. This is standard, using a parametrization of E by an appropriate Shimura curve. See [How04c, Prop. 1.2.1], for example. \square

The anticyclotomic \mathbf{Z}_p -extension K_∞^{ac}/K is contained in $\tilde{K}_\infty = \bigcup_{k \geq 0} K[p^k]$, and $\text{Gal}(\tilde{K}_\infty/K)$ is isomorphic to $\Gamma^{\text{ac}} \times \Delta$, with Δ a finite abelian group. For every integer $S \geq 1$ coprime to Np and every $n \geq 0$, we let $K_n^{\text{ac}}[S]$ denote the compositum $K_n^{\text{ac}}K[S]$, and define the points $P_n[S] \in E(K_n^{\text{ac}}[S])$ by

$$P_n[S] := \text{tr}_{K_n^{\text{ac}}[S]}^{K[S^{p^{k(n)}}]}(P[S^{p^{k(n)}}]),$$

where $k(n) = \min\{k \geq 0 \mid K_n^{\text{ac}} \subseteq K[p^k]\}$. Let $z_n[S]$ be the image of $P_n[S]$ under the Kummer map

$$E(K_n^{\text{ac}}[S]) \otimes \mathbf{Z}_p \longrightarrow H^1(K_n^{\text{ac}}[S], T),$$

and denote by cor_n^{n+1} the corestriction map for the extension $K_{n+1}^{\text{ac}}[S]/K_n^{\text{ac}}[S]$. Then from the norm-compatibility in Proposition 4.1 it follows that

$$(4.1) \quad \text{cor}_n^{n+1}(z_{n+1}[S]) = -z_{n-1}[S],$$

since $a_p = 0$.

Lemma 4.2. *The classes $z_n[S]$ lie in the image of the natural map*

$$H^1(K[S], \mathbf{T}^{\text{ac}}) \longrightarrow H^1(K_n^{\text{ac}}[S], T).$$

Proof. The obvious long exact sequence shows that the cokernel of the map in the statement is controlled by

$$H^2(K[S], \mathbf{T}^{\text{ac}})[\omega_n].$$

Since $H^2(K[S], \mathbf{T}^{\text{ac}})$ is finitely generated over Λ^{ac} , the above module stabilizes for $n \gg 0$. On the other hand, from the norm-relation (4.1) we immediately see that

$$\frac{\omega_{n'}^\epsilon}{\omega_n^\epsilon} z_{n'}[S] = \pm z_n[S]$$

for all $n' > n$ with the same parity, where $\epsilon = (-1)^n$. Letting n' approach infinity, it follows that any $z_n[S]$ must have zero image in the above cokernel, hence the result. \square

The following freeness result will be a key ingredient in the construction of the plus/minus Heegner classes.

Lemma 4.3. *Assume that $E[p]|_{G_K}$ is irreducible. Then $H^1(K[S], \mathbf{T}^{\text{ac}})$ is free over Λ^{ac} .*

Proof. Let $M_S := H^1(K[S], \mathbf{T}^{\text{ac}})$, and identify $\Lambda^{\text{ac}} \simeq \mathbf{Z}_p[[X]]$ setting $\gamma^{\text{ac}} - 1 = X$. We first claim that the two maps

$$\alpha : M_S \xrightarrow{X} M_S, \quad \beta : M_S/XM_S \xrightarrow{p} M_S/XM_S$$

are injective. Indeed, the irreducibility assumption on $E[p]|_{G_K}$ implies that $T^{G_{K^{\text{ac}}}} = \{0\}$, and hence the injectivity of α follows from [PR00, §1.3.3]. We thus get an injection $M_S/XM_S \hookrightarrow H^1(K[S], T)$, and so to establish the injectivity of β it suffices to show the injectivity of the map

$$H^1(K[S], T) \xrightarrow{p} H^1(K[S], T),$$

but this follows again from the G_K -irreducibility of $E[p]$. By the structure theorem for finitely generated modules over Λ^{ac} , the above shows that M_S injects with finite cokernel N into a free module of finite rank. If $N \neq 0$, then $\text{Tor}_1^{\Lambda^{\text{ac}}}(N, \Lambda^{\text{ac}}/X\Lambda^{\text{ac}})$ is a nonzero \mathbf{Z}_p -torsion injecting into M_S/XM_S , contradicting the injectivity of β . Hence $N = 0$ and M_S is free over Λ^{ac} . \square

Let ϵ denote the sign $(-1)^n$, and set $\omega_n^{\pm} := \omega_n^{\pm}((1+Y)^{p^a} - 1)$ to lighten the notation. A straightforward induction argument (cf. [DI08, Lemma 4.2]) using the compatibility (4.1) then shows that

$$\omega_n^{\epsilon} z_n[S] = 0$$

By Lemma 4.2 and the freeness result of Lemma 4.3, this implies that there is a unique class

$$z_n[S]^{\epsilon} \in H^1(K_n^{\text{ac}}[S], T)/\omega_n^{\epsilon} H^1(K_n[S], T)$$

such that

$$\tilde{\omega}_n^{-\epsilon} z_n[S]^{\epsilon} = (-1)^{\lfloor \frac{n+1}{2} \rfloor} z_n[S].$$

Lemma 4.4. *For each $\epsilon \in \{\pm 1\}$ the sequences $\{z_n[S]^{\epsilon}\}_{n \equiv \epsilon \pmod{2}}$ are compatible under the natural projections*

$$H^1(K_n^{\text{ac}}[S], T)/\omega_n^{\epsilon} H^1(K_n^{\text{ac}}[S], T) \longrightarrow H^1(K_{n-2}^{\text{ac}}[S], T)/\omega_{n-2}^{\epsilon} H^1(K_{n-2}^{\text{ac}}[S], T)$$

induced by corestriction.

Proof. In light of the freeness result of Lemma 4.3, the argument in [DI08, Lemma 2.9] applies verbatim. \square

We may thus define

$$(4.2) \quad \mathbf{z}[S]^{\pm} := \varprojlim_n z_n[S]^{\pm}$$

as an element in $H^1(K[S], \mathbf{T}^{\text{ac}})$, where the limit is over n with the fixed parity determined by the sign \pm . Note that since $\{\omega_n^{\pm}\}_n$ forms a basis for the topology of Λ^{ac} , the class $\mathbf{z}[S]^{\pm}$ is well-defined.

4.2. Explicit reciprocity law. Similarly as in [CH15], the classes $\mathbf{z}^{\pm} := \mathbf{z}[1]^{\pm}$ satisfy an explicit reciprocity law relating them to some of the anticyclotomic p -adic L -functions in §1.3.

Recall from §2.1 the element $d = \{d_m\}_m \in \varprojlim_m \mathcal{O}_{k_m}^{\times}$ generating this $\mathbf{Z}_p[[U]]$ -module, and let

$$F_{d,2} = \varprojlim_m \sum_{\sigma \in U/p^m U} d_m^{\sigma} \cdot \sigma^2,$$

viewed as an element in Λ_{R_0} . By the discussion in [LZ14, §6.4] on Katz p -adic L -functions (see also [loc.cit., §3.2]), the quotient $\mathcal{L}_p^{\text{Katz}}(K)/F_{d,2}$ gives rise to a nonzero element in Λ^{ac} , rather than just $\Lambda_{R_0}^{\text{ac}}$.

Lemma 4.5. *We have the equality*

$$F_{d,2} = \left(\varprojlim_m \sum_{\sigma \in U/p^m U} d_m^\sigma \cdot \sigma \right)^2$$

up to a unit in $\mathbf{Z}_p[[U]]^\times$.

Proof. This follows from a straightforward calculation. \square

Thus the p -adic L -function $\mathcal{L}_p(f/K) \in \Lambda_{R_0}$ of Theorem 1.11 can be written as the product

$$(4.3) \quad \mathcal{L}_p(f/K) = \mathcal{L}_p(f/K) \cdot F_d^2 \cdot U,$$

for some $\mathcal{L}_p(f/K) \in \Lambda$ and $U \in \Lambda^\times$, where $F_d = \varprojlim_m \sum_{\sigma \in U/p^m U} d_m^\sigma \cdot \sigma$. Moreover, letting $\mathcal{L}_{p,\text{ac}}(f/K)$ denote the image of $\mathcal{L}_p(f/K)$ under the natural projection $\Lambda \rightarrow \Lambda^{\text{ac}}$, we see from Corollary 1.12 that

$$(4.4) \quad \mathcal{L}_p^{\text{BDP}}(f/K)^2 = \mathcal{L}_{p,\text{ac}}(f/K) \cdot F_d^2 \cdot U'$$

for some $U' \in (\Lambda^{\text{ac}})^\times$. For the following key formula, note that the classes \mathbf{z}^\pm satisfy $\text{loc}_v(\mathbf{z}^\pm) \in H_\pm^1(K, \mathbf{T}^{\text{ac}})$ for all primes v above p , and so we may consider the image of $\text{loc}_v(\mathbf{z}^\pm)$ under the plus/minus logarithm maps $\text{Log}_{\text{ac}}^\pm$ constructed in §2.3.

Theorem 4.6 (Explicit reciprocity law). *With notations as above, we have the equality*

$$(4.5) \quad \text{Log}_{\text{ac}}^\pm(\text{loc}_p(\mathbf{z}^\pm)) = \mathcal{L}_p^{\text{BDP}}(f/K) \cdot F_d \cdot \sigma_{-1,p},$$

where $\sigma_{-1,p} := \text{rec}_p(-1)|_{K_\infty^{\text{ac}}} \in \Gamma^{\text{ac}}$. In particular, the class \mathbf{z}^\pm is non-torsion.

Proof. We give the proof for \mathbf{z}^+ , the proof for the other sign being virtually the same. Let ψ be an anticyclotomic Hecke character of infinity type $(1, -1)$ and conductor prime to p , and let $\mathcal{L}_{p,\psi}(f) \in \Lambda_{R_0}^{\text{ac}}$ be as in [CH15, Def. 3.5]. The p -adic L -function $\mathcal{L}_p^{\text{BDP}}(f/K)$ of Theorem 1.7 is then given by

$$\mathcal{L}_p^{\text{BDP}}(f/K) = \text{Tw}_{\psi^{-1}}(\mathcal{L}_{p,\psi}(f)),$$

where $\text{Tw}_{\psi^{-1}} : \Lambda_{R_0}^{\text{ac}} \rightarrow \Lambda_{R_0}^{\text{ac}}$ is the R_0 -linear isomorphism given by $\gamma \mapsto \psi^{-1}(\gamma)\gamma$ for $\gamma \in \Gamma^{\text{ac}}$. Now let $\phi : \Gamma^{\text{ac}} \rightarrow \mu_{p^\infty}$ be a non-trivial finite order character, let $n > 0$ be the smallest positive integer such that ϕ factors through $\Gamma^{\text{ac}}/p^n \Gamma^{\text{ac}}$, and assume that n is even. Following the calculations in [CH15, Thm. 4.8], we then find that

$$\begin{aligned} \mathcal{L}_p^{\text{BDP}}(f/K)(\phi^{-1}) &= \mathfrak{g}(\phi^{-1})\phi(p^n)p^{-n} \sum_{\sigma \in \Gamma^{\text{ac}}/p^n \Gamma^{\text{ac}}} \phi(\sigma) \log_{\hat{E}}(\sigma P[p^n]) \\ &= \phi(-1) \cdot \frac{\phi(p^n)}{\mathfrak{g}(\phi)} \cdot (-1)^{n/2} \tilde{\omega}_n^-(\phi) \sum_{\sigma \in \Gamma^{\text{ac}}/p^n \Gamma^{\text{ac}}} \phi(\sigma) \log_{\hat{E}}(\sigma P[p^n]^+), \end{aligned}$$

where we used the definition of $P[p^n]^+$ for the second equality. Combined with the interpolation properties of the map Log^+ (see Lemma 2.4), this shows that

$$\begin{aligned} \mathcal{L}_p^{\text{BDP}}(f/K)(\phi^{-1}) &= \phi(-1) \cdot \frac{\phi(p^n)}{\mathfrak{g}(\phi)} \sum_{\sigma \in \Gamma^{\text{ac}}/p^n \Gamma^{\text{ac}}} \phi(\sigma) \log_{\hat{E}}(c_n^\sigma) \cdot \text{Log}_{\text{ac}}^+(\text{loc}_p(\mathbf{z}^+))(\phi^{-1}) \\ (4.6) \quad &= \phi(-1) \sum_{\sigma \in \Gamma^{\text{ac}}/p^n \Gamma^{\text{ac}}} \phi(\sigma) d_{n+a}^\sigma \cdot \text{Log}_{\text{ac}}^+(\text{loc}_p(\mathbf{z}^+))(\phi^{-1}). \end{aligned}$$

Letting ϕ vary, equality (4.5) follows immediately from (4.6). Finally, the non-triviality of \mathbf{z}^\pm follows from the explicit reciprocity law (4.5) combined with the nonvanishing of $\mathcal{L}_p^{\text{BDP}}(f/K)$ in Theorem 1.7. \square

Remark 4.7. That the class \mathbf{z}^\pm in non-torsion over Λ^{ac} can also be deduced from the proof by Cornut–Vatsal of Mazur’s conjecture for Heegner points (see [CV07, Thm. 1.5] and the discussion right after it). However, the local strengthening of this non-triviality provided by Theorem 4.6 will be a vital ingredient for our main results in this paper.

4.3. Anticyclotomic main conjectures. Motivated by Perrin-Riou’s Heegner point main conjecture in the ordinary setting [PR87a], it is natural to formulate the following two variants (one for each choice of signs) for supersingular primes.

Conjecture 4.8. *Both $X^{\pm,\pm}(K, \mathbf{A}^{\text{ac}})$ and $\text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}})$ have Λ^{ac} -rank 1, and*

$$Ch_{\Lambda^{\text{ac}}}(X^{\pm,\pm}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}) = Ch_{\Lambda^{\text{ac}}}\left(\frac{\text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^\pm}\right)^2.$$

On the other hand, the general philosophy of Iwasawa–Greenberg main conjectures predicts the following for the anticyclotomic p -adic L -function $\mathcal{L}_p^{\text{BDP}}(f/K) \in \Lambda_{R_0}^{\text{ac}}$ of Theorem 1.7. Note that by (4.3) and (4.4), there exists an element $\mathcal{L}_p^{\text{BDP}}(f/K) \in \Lambda^{\text{ac}}$ such that

$$(4.7) \quad (\mathcal{L}_p^{\text{BDP}}(f/K)^2) = (\mathcal{L}_p^{\text{BDP}}(f/K)^2) = (\mathcal{L}_{p,\text{ac}}(f/K))$$

as principal ideals of $\Lambda_{R_0}^{\text{ac}}$.

Conjecture 4.9. *$X^{\text{rel},\text{str}}(K, \mathbf{A}^{\text{ac}})$ is Λ^{ac} -torsion, and*

$$Ch_{\Lambda^{\text{ac}}}(X^{\text{rel},\text{str}}(K, \mathbf{A}^{\text{ac}})) = (\mathcal{L}_p^{\text{BDP}}(f/K)^2).$$

As we shall show, these two conjectures are intimately related. In fact, in Section 5.1 we will reduce the proof of Conjecture 4.9 to the proof of Conjecture 4.8 (which we prove here). The reduction relies crucially on the explicit reciprocity law of Theorem 4.6, as exploited in some of the next three lemmas.

Lemma 4.10. *Assume that $\text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}})$ has Λ^{ac} -rank 1. Then*

$$(4.8) \quad \text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}}) = \text{Sel}^{\pm,\text{rel}}(K, \mathbf{T}^{\text{ac}})$$

and $X^{\pm,\text{str}}(K, \mathbf{A}^{\text{ac}})$ is a torsion Λ^{ac} -module.

Proof. Consider the exact sequence

$$(4.9) \quad 0 \longrightarrow \text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}}) \longrightarrow \text{Sel}^{\pm,\text{rel}}(K, \mathbf{T}^{\text{ac}}) \xrightarrow{\text{loc}_{\overline{\mathfrak{p}}}} \frac{H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})}{H_{\pm}^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})} \longrightarrow \text{coker}(\text{loc}_{\overline{\mathfrak{p}}}) \longrightarrow 0.$$

Since both the first and the third terms in this sequence have Λ^{ac} -rank 1, we immediately see that $\text{rank}_{\Lambda^{\text{ac}}} \text{Sel}^{\pm,\text{rel}}(K, \mathbf{T}^{\text{ac}}) = 1$, and that the image of the map $\text{loc}_{\overline{\mathfrak{p}}}$ is Λ^{ac} -torsion. Moreover, since $H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})/H_{\pm}^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})$ is torsion-free, this shows that $\text{loc}_{\overline{\mathfrak{p}}}$ is the zero map, from where equality (4.8) follows. Thus (4.9) and global duality yields the exact sequence

$$0 \longrightarrow \frac{H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})}{H_{\pm}^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\text{ac}})} \longrightarrow X^{\pm,\pm}(K, \mathbf{A}^{\text{ac}}) \longrightarrow X^{\pm,\text{str}}(K, \mathbf{A}^{\text{ac}}) \longrightarrow 0,$$

and since the first two terms in this sequence have Λ^{ac} -rank 1 (using Lemma 2.7(1) for the second term), the lemma follows. \square

Lemma 4.11. *Assume that $\text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}})$ has Λ^{ac} -rank 1. Then for any height one prime \mathfrak{p} of Λ^{ac} we have*

$$\text{ord}_{\mathfrak{p}}(\mathcal{L}_p^{\text{BDP}}(f/K)) = \text{length}_{\mathfrak{p}}(\text{coker}(\text{loc}_{\mathfrak{p}})) + \text{length}_{\mathfrak{p}}\left(\frac{\text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^\pm}\right),$$

where $\text{loc}_{\mathfrak{p}} : \text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}}) \rightarrow H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{T}^{\text{ac}})$ is the natural restriction map.

Proof. Consider the tautological exact sequence

$$(4.10) \quad 0 \longrightarrow \mathrm{Sel}^{\mathrm{str},\pm}(K, \mathbf{T}^{\mathrm{ac}}) \longrightarrow \mathrm{Sel}^{\pm,\pm}(K, \mathbf{T}^{\mathrm{ac}}) \longrightarrow H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}}) \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}) \longrightarrow 0.$$

The nonvanishing of $\mathcal{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K)$ (and hence of $\mathcal{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K)$) together with Theorem 4.6 imply that the image of $\mathbf{z}^{\pm} \in \mathrm{Sel}^{\pm,\pm}(K, \mathbf{T}^{\mathrm{ac}})$ under the map $\mathrm{loc}_{\mathfrak{p}}$ is not Λ^{ac} -torsion. Since $\mathrm{Sel}^{\pm,\pm}(K, \mathbf{T}^{\mathrm{ac}})$ has Λ^{ac} -rank 1 by assumption, this shows that $\mathrm{Sel}^{\mathrm{str},\pm}(K, \mathbf{T}^{\mathrm{ac}})$ is Λ^{ac} -torsion, and since $H^1(K, \mathbf{T}^{\mathrm{ac}})$ is Λ^{ac} -torsion-free (see e.g. [How04b, Lemma 2.2.9]), it follows that

$$(4.11) \quad \mathrm{Sel}^{\mathrm{str},\pm}(K, \mathbf{T}^{\mathrm{ac}}) = \{0\}.$$

From (4.10) we thus deduce the exact sequence

$$0 \longrightarrow \frac{\mathrm{Sel}^{\pm,\pm}(K, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathbf{z}^{\pm}} \longrightarrow \frac{H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathrm{loc}_{\mathfrak{p}}(\mathbf{z}^{\pm})} \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}) \longrightarrow 0,$$

and since by the explicit reciprocity law of Theorem 4.6 the map $\mathrm{Log}_{\mathrm{ac}}^{\pm}$ induces a Λ^{ac} -module isomorphism

$$\frac{H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathrm{loc}_{\mathfrak{p}}(\mathbf{z}^{\pm})} \xrightarrow{\sim} \frac{\Lambda^{\mathrm{ac}}}{\Lambda^{\mathrm{ac}} \cdot \mathcal{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K)},$$

the result follows. \square

Lemma 4.12. *Assume that $\mathrm{Sel}^{\pm,\pm}(K, \mathbf{T}^{\mathrm{ac}})$ has Λ^{ac} rank 1. Then the module $X^{\mathrm{rel},\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac} -torsion, and for any height one prime \mathfrak{P} of Λ^{ac} with $\mathfrak{P} \neq p\Lambda^{\mathrm{ac}}$ we have*

$$\mathrm{length}_{\mathfrak{P}}(X^{\mathrm{rel},\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}})) = \mathrm{length}_{\mathfrak{P}}(X^{\pm,\pm}(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) + 2 \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})),$$

where $\mathrm{loc}_{\mathfrak{p}} : \mathrm{Sel}^{\pm,\pm}(K, \mathbf{T}^{\mathrm{ac}}) \rightarrow H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}})$ is the natural restriction map.

Proof. Global duality yields the exact sequence

$$(4.12) \quad 0 \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}) \longrightarrow X^{\mathrm{rel},\pm}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow X^{\pm,\pm}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow 0.$$

As shown in the proof of Lemma 4.11, the first term in the sequence is Λ^{ac} -torsion; since by Lemma 2.7(1) the assumption implies that $X^{\pm,\pm}(K, \mathbf{A}^{\mathrm{ac}})$ has Λ^{ac} -rank 1, this shows that the same is true for $X^{\mathrm{rel},\pm}(K, \mathbf{A}^{\mathrm{ac}})$, and by Lemma 2.7(2) it follows that $X^{\mathrm{str},\pm}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac} -torsion. Thus taking Λ^{ac} -torsion in (4.12) and using Lemma 2.7(3), it follows that

$$(4.13) \quad \mathrm{length}_{\mathfrak{P}}(X^{\mathrm{str},\pm}(K, \mathbf{A}^{\mathrm{ac}})) = \mathrm{length}_{\mathfrak{P}}(X^{\pm,\pm}(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) + \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}))$$

for any height one prime \mathfrak{P} of Λ^{ac} different from $p\Lambda^{\mathrm{ac}}$.

Another application of global duality yields the exact sequence

$$(4.14) \quad 0 \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}^{\mathrm{rel}}) \longrightarrow X^{\mathrm{rel},\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow X^{\pm,\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow 0,$$

where $\mathrm{loc}_{\mathfrak{p}}^{\mathrm{rel}} : \mathrm{Sel}^{\pm,\mathrm{rel}}(K, \mathbf{T}^{\mathrm{ac}}) \rightarrow H_{\pm}^1(K_{\mathfrak{p}}, \mathbf{T}^{\mathrm{ac}})$ is the natural restriction map. By Lemma 4.11, this is the same as the map $\mathrm{loc}_{\mathfrak{p}}$ in the statement, and hence $\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}^{\mathrm{rel}}) = \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})$ is Λ^{ac} -torsion. Since $X^{\pm,\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac} -torsion by Lemma 4.10, we conclude from (4.14) that $X^{\mathrm{rel},\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac} -torsion. Combining (4.14) and (4.13), we thus have

$$\begin{aligned} \mathrm{length}_{\mathfrak{P}}(X^{\mathrm{rel},\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}})) &= \mathrm{length}_{\mathfrak{P}}(X^{\pm,\mathrm{str}}(K, \mathbf{A}^{\mathrm{ac}})) + \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})) \\ &= \mathrm{length}_{\mathfrak{P}}(X^{\pm,\pm}(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) + 2 \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})) \end{aligned}$$

for any height one prime \mathfrak{P} of Λ^{ac} with $\mathfrak{P} \neq p\Lambda^{\mathrm{ac}}$, as was to be shown. \square

4.4. Kolyvagin system argument. The purpose of this section is to prove the following result, establishing under mild assumptions one of the divisibility predicted by Conjecture 4.8.

Theorem 4.13. *Assume that $\text{Gal}(\overline{\mathbf{Q}}/K) \rightarrow \text{Aut}_{\mathbf{Z}_p}(T)$ is surjective. Then both $X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})$ and $\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})$ have Λ^{ac} -rank 1, and*

$$Ch_{\Lambda^{\text{ac}}}(X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}) \supseteq Ch_{\Lambda^{\text{ac}}}\left(\frac{\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^{\pm}}\right)^2.$$

For the proof of Theorem 4.13, we will adapt to our supersingular setting the Kolyvagin system techniques developed by Howard [How04b] in the ordinary case. More precisely, we will use the classes $P[S]^{\pm}$ introduced in (4.2) to build a certain Kolyvagin system for \mathbf{T}^{ac} ; the nontriviality of this system will follow from the nontriviality of \mathbf{z}^{\pm} established in Theorem 4.6, and Theorem 4.13 will then follow from a suitable adaptation of Howard's arguments.

As in [How04b, §1.1], by a Selmer structure on \mathbf{T}^{ac} we mean a choice of a local condition $H_{\mathcal{F}}^1(K_v, \mathbf{T}^{\text{ac}}) \subseteq H^1(K_v, \mathbf{T}^{\text{ac}})$ for each place $v \in \Sigma$. (Here Σ is any finite set of places of K containing those above p , those above ∞ , and those where V ramifies.) We define the Selmer structure \mathcal{F}^{\pm} on \mathbf{T}^{ac} to be the unramified local condition at the places in Σ not dividing p , and the plus/minus local condition $H_{\pm}^1(K_v, \mathbf{T}^{\text{ac}})$ at the primes v above p .

For the statement of the next result, we refer the reader to [How04b, §1.2] for the definition of the module of Kolyvagin systems $\mathbf{KS}(\mathbf{T}^{\text{ac}}, \mathcal{F}, \mathcal{L})$ attached to a Selmer structure \mathcal{F} on \mathbf{T}^{ac} and a certain set \mathcal{L} of primes inert in K .

Theorem 4.14. *There exists a Kolyvagin system $\kappa^{\pm} \in \mathbf{KS}(\mathbf{T}^{\text{ac}}, \mathcal{F}^{\pm}, \mathcal{L})$ with $\kappa_1^{\pm} = \mathbf{z}^{\pm}$.*

Proof. Let \mathcal{L}_0 be the set of rational primes ℓ not dividing pN and inert in K . For each $\ell \in \mathcal{L}_0$, let λ be the prime of K above ℓ , and denote by I_{ℓ} the smallest ideal of Λ^{ac} containing $\ell + 1$ for which the Frobenius element $\text{Fr}_{\lambda} \in G_{K_{\lambda}}$ acts trivially on $\mathbf{T}^{\text{ac}}/I_{\ell}\mathbf{T}^{\text{ac}}$. Let $\mathcal{L} = \mathcal{L}(\mathbf{T}^{\text{ac}}) \subseteq \mathcal{L}_0$ consist of the primes $\ell \in \mathcal{L}_0$ with $I_{\ell} \subseteq p\mathbf{Z}_p$, and let \mathcal{N} be the set of square-free products S of primes in \mathcal{L} , with the convention that $1 \in \mathcal{N}$. For each $S \in \mathcal{N}$, define $I_S = \sum_{\ell|S} I_{\ell}$.

Now from the plus/minus Heegner classes $\mathbf{z}[S]^{\pm}$ defined in (4.2), the derivative construction in [How04b, §1.7] produces classes

$$\kappa_S^{\pm} \in H^1(K, \mathbf{T}^{\text{ac}}/I_S\mathbf{T}^{\text{ac}}),$$

indexed by the products $S \in \mathcal{N}$, with $\kappa_1^{\pm} = \mathbf{z}[1]^{\pm} = \mathbf{z}^{\pm}$. The verification that these classes form a Kolyvagin system for the Selmer structure \mathcal{F}^{\pm} on \mathbf{T}^{ac} follows from the same argument as in [How04b, Lemma 2.3.4], the only difference being at the places $v|p$, where we are led to show that the localization of κ_S^{\pm} at v is contained in $H_{\mathcal{F}^{\pm}}^1(K_v, \mathbf{T}^{\text{ac}}/I_S\mathbf{T}^{\text{ac}})$, defined as the image of the natural map

$$H_{\mathcal{F}^{\pm}}^1(K_v, \mathbf{T}^{\text{ac}}) \longrightarrow H^1(K_v, \mathbf{T}^{\text{ac}}/I_S\mathbf{T}^{\text{ac}}).$$

But this follows from the same argument as in the proof of Lemma 4.3. \square

Let \mathfrak{P} be a height one prime of Λ^{ac} , let $S_{\mathfrak{P}}$ denote the integral closure of $\Lambda^{\text{ac}}/\mathfrak{P}$, and let $\varpi_{\mathfrak{P}} \in S_{\mathfrak{P}}$ be a uniformizer. Define the Galois representations

$$T_{\mathfrak{P}} := \mathbf{T}^{\text{ac}} \otimes_{\Lambda^{\text{ac}}} S_{\mathfrak{P}}, \quad A_{\mathfrak{P}} := \mathbf{A}^{\text{ac}} \otimes_{\Lambda^{\text{ac}}} S_{\mathfrak{P}},$$

and let $T_{\mathfrak{P},m}$ and $A_{\mathfrak{P},m}$ be their reduction modulo $\varpi_{\mathfrak{P}}^m$.

For each place $v|p$ in K , define $H_{\pm}^1(K_v, T_{\mathfrak{P}}) \subseteq H_{\pm}^1(K_v, T_{\mathfrak{P}})$ to be the image of the natural map

$$H^1(K_v, \mathbf{T}^{\text{ac}}) \longrightarrow H^1(K_v, \mathbf{T}^{\text{ac}} \otimes_{\Lambda^{\text{ac}}} \Lambda^{\text{ac}}/\mathfrak{P}) \longrightarrow H^1(K_v, T_{\mathfrak{P}}),$$

and define $H_{\pm}^1(K_v, T_{\mathfrak{P},m}) \subseteq H^1(K_v, T_{\mathfrak{P},m})$ in the same manner. Also, let $H_{\pm}^1(K_v, A_{\mathfrak{P}})$ (resp. $H_{\pm}^1(K_v, A_{\mathfrak{P},m})$) be the orthogonal complement of $H_{\pm}^1(K_v, T_{\mathfrak{P}})$ (resp. $H_{\pm}^1(K_v, T_{\mathfrak{P},m})$) under the local Tate pairing.

Lemma 4.15. *For every height one prime $\mathfrak{P} \subseteq \Lambda^{\text{ac}}$ with $\mathfrak{P} \neq p\Lambda^{\text{ac}}$, and for every place v of K , the natural maps*

$$\begin{aligned} H_{\mathcal{F}^\pm}^1(K_v, \mathbf{T}^{\text{ac}}/\mathfrak{P}\mathbf{T}^{\text{ac}}) &\longrightarrow H_{\mathcal{F}_{\mathfrak{P}}^\pm}^1(K_v, T_{\mathfrak{P}}), \\ H_{\mathcal{F}_{\mathfrak{P}}^\pm}^1(K_v, A_{\mathfrak{P}}) &\longrightarrow H_{\mathcal{F}^\pm}^1(K_v, \mathbf{A}^{\text{ac}}[\mathfrak{P}]) \end{aligned}$$

have finite kernel and cokernel of order bounded by a constant depending only on $[S_{\mathfrak{P}} : \Lambda^{\text{ac}}/\mathfrak{P}]$.

Proof. Let m be any positive integer, and $n \gg 0$ be such that we have the inclusion of ideals of Λ^{ac} :

$$(\omega_n(X), p^m) \subseteq (\mathfrak{P}, p^m).$$

Then it follows from [Kim07, Prop. 3.14] (cf. [loc.cit., Prop. 4.11]) that for each $v|p$ in K , the module $H_{\pm}^1(K_v, T_{\mathfrak{P},m})$ is the exact annihilator of $H_{\pm}^1(K_{\overline{v}}, T_{\mathfrak{P},m})$ under local Tate duality. In particular, $H_{\pm}^1(K_v, A_{\mathfrak{P},m})$ can be identified with $H_{\pm}^1(K_{\overline{v}}, T_{\mathfrak{P},m})$. Hence by [Kim07, Prop. 4.18] the second map in the statement has kernel and cokernel with the required bounds, and taking duals the same properties for the first map follow. \square

Proof of Theorem 4.13. We can now adapt the argument in the proof [How04b, Thm. 2.2.10]. Indeed, for every height one prime $\mathfrak{P} \subseteq \Lambda^{\text{ac}}$ with $\mathfrak{P} \neq p\Lambda^{\text{ac}}$, we have a map

$$\mathbf{KS}(\mathbf{T}^{\text{ac}}, \mathcal{F}^\pm, \mathcal{L}(\mathbf{T}^{\text{ac}})) \longrightarrow \mathbf{KS}(T_{\mathfrak{P}}, \mathcal{F}_{\mathfrak{P}}^\pm, \mathcal{L}(T_{\mathfrak{P}})),$$

where $\mathcal{F}_{\mathfrak{P}}^\pm$ is the Selmer structure on $T_{\mathfrak{P}}$ naturally induced from \mathcal{F}^\pm . Letting $\kappa^\pm(\mathfrak{P})$ be the image of the Kolyvagin system κ^\pm of Theorem 4.14 under this map, it follows from Theorem 4.6 and Lemma 4.15 that $\kappa_1^\pm(\mathfrak{P})$ generates and infinite $S_{\mathfrak{P}}$ -submodule of $H_{\mathcal{F}_{\mathfrak{P}}^\pm}^1(K, T_{\mathfrak{P}})$ for all but finitely many \mathfrak{P} . To deduce, as in [How04b, Prop. 2.1.3], that for any such \mathfrak{P} the module $H_{\mathcal{F}_{\mathfrak{P}}^\pm}^1(K, T_{\mathfrak{P}})$ is free of rank one over $S_{\mathfrak{P}}$, it suffices to show that the triple $(T_{\mathfrak{P}}, \mathcal{F}_{\mathfrak{P}}^\pm, \mathcal{L}(T_{\mathfrak{P}}))$ satisfies the hypotheses (H.0)–(H.5) of [loc.cit., §1.2]. The only difference here with respect to the verification of these hypotheses in [How04b, Prop. 2.1.3] is the self-duality condition in hypothesis (H.4), but this follows from [Kim07, Prop. 3.14] as indicated above.

By Lemma 4.15, this shows that $H_{\mathcal{F}^\pm}^1(K, \mathbf{T}^{\text{ac}}) \otimes_{\Lambda^{\text{ac}}} S_{\mathfrak{P}}$ is a free $S_{\mathfrak{P}}$ -module of rank one, from where the first part of Theorem 4.13 follows immediately, and for the second part the argument in [How04b, Thm. 2.2.10] applies verbatim. \square

5. END OF PROOFS

5.1. Proof of the main conjectures. For the ease of notation, set

$$X_{\mathfrak{p}}(K, \mathbf{A}) := X^{\text{rel, str}}(K, \mathbf{A}),$$

and similarly for $X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})$. The next two results (Theorems 5.1 and 5.2 below) correspond to Theorems C and D in the Introduction. Let N^- denote the product of the prime factors of N which are non-split in K .

Theorem 5.1. *Let E/\mathbf{Q} be an elliptic curve of conductor N with good supersingular reduction at p , and let K/\mathbf{Q} be an imaginary quadratic field of discriminant prime to N . Assume that the triple (E, p, K) satisfy the following:*

- $p \geq 5$,
- $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K ,
- hypothesis (Heeg) holds,
- N is square-free,
- $N^- \neq 1$,
- $E[p]$ is ramified at every prime $\ell|N^-$,
- $\text{Gal}(\overline{\mathbf{Q}}/K) \rightarrow \text{Aut}_{\mathbf{Z}_p}(T_p(E))$ is surjective.

Then:

- (1) Both $X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})$ and $\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})$ have Λ^{ac} -rank 1, and

$$Ch_{\Lambda^{\text{ac}}}(X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}) = Ch_{\Lambda^{\text{ac}}}\left(\frac{\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^{\pm}}\right)^2$$

up to powers of $p\Lambda^{\text{ac}}$.

- (2) $X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})$ is Λ^{ac} -torsion, and

$$Ch_{\Lambda^{\text{ac}}}(X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})) = (\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)^2)$$

as ideals in Λ^{ac} .

Proof. By Theorem 4.13 we know that $\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})$ has Λ^{ac} -rank 1. By Lemma 2.7(1) the same is true for $X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})$, and by Lemma 4.12 the module $X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})$ is Λ^{ac} -torsion. Let \mathfrak{P} be a height one prime of Λ^{ac} . Then by the divisibility in Theorem 4.13 we have

$$(5.1) \quad \text{length}_{\mathfrak{P}}(X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}) \leq 2 \text{length}_{\mathfrak{P}}\left(\frac{\text{Sel}^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^{\pm}}\right).$$

Combined with Lemma 4.12 and Lemma 4.11, respectively, this implies that

$$\begin{aligned} \text{length}_{\mathfrak{P}}(X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})) &\leq 2 \text{length}_{\mathfrak{P}}\left(\frac{\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^{\pm}}\right) + 2 \text{length}_{\mathfrak{P}}(\text{coker}(\text{loc}_{\mathfrak{p}})) \\ &= 2 \text{ord}_{\mathfrak{P}}(\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)), \end{aligned}$$

provided $\mathfrak{P} \neq p\Lambda^{\text{ac}}$ for the first equality. Therefore

$$(5.2) \quad Ch_{\Lambda^{\text{ac}}}(X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})) \supseteq (\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)^2)$$

up to powers of $p\Lambda^{\text{ac}}$. It remains to show the two divisibilities \subseteq in the theorem. Let I^{cyc} be the principal ideal of Λ generated by $\gamma^{\text{cyc}} - 1$. By [Wan14, Prop. 3.1] (whose proof does not rely on any p -ordinarity hypothesis on E), the natural restriction map $H^1(K_{\infty}^{\text{ac}}, E[p^{\infty}]) \rightarrow H^1(K_{\infty}, E[p^{\infty}])$ induces an isomorphism

$$(5.3) \quad X_{\mathfrak{p}}(K, \mathbf{A})/I^{\text{cyc}} X_{\mathfrak{p}}(K, \mathbf{A}) \simeq X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})$$

as $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ -modules. Combined with (4.4), the two-variable divisibility in [Wan15, Thm. 6.13] thus yields the divisibility

$$Ch_{\Lambda^{\text{ac}}}(X_{\mathfrak{p}}(K, \mathbf{A}^{\text{ac}})) \subseteq (\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)^2)$$

as fractional ideals of $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. In particular, equality in (5.2) holds as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and since $\mu(\mathcal{L}_{\mathfrak{p}}^{\text{BDP}}(f/K)) = 0$ by Theorem 1.8, the equality holds integrally. Combined with Lemmas 4.11 and 4.12, this implies that for any height one prime $\mathfrak{P} \subseteq \Lambda^{\text{ac}}$ with $\mathfrak{P} \neq p\Lambda^{\text{ac}}$, equality in (5.1) holds, and the proof is complete. \square

Theorem 5.2. *Under the hypotheses of Theorem 5.1, the following hold:*

- (1) $X^{\pm, \text{str}}(K, \mathbf{A})$ is Λ -torsion, $\text{Sel}^{\pm, \text{rel}}(K, \mathbf{T})$ has Λ -rank 1, and

$$Ch_{\Lambda}(X^{\pm, \text{str}}(K, \mathbf{A})) \cdot \mathcal{H}^{\pm} = Ch_{\Lambda}\left(\frac{\text{Sel}^{\pm, \text{rel}}(K, \mathbf{T})}{\Lambda \cdot \mathcal{BF}^{\pm}}\right)$$

as ideals in Λ .

- (2) $X_{\mathfrak{p}}(K, \mathbf{A})$ is Λ -torsion, and

$$Ch_{\Lambda}(X_{\mathfrak{p}}(K, \mathbf{A})) = (\mathcal{L}_{\mathfrak{p}}(f/K))$$

as ideals in Λ .

Proof. Given the equalities in the anticyclotomic main conjecture established in Theorem 5.1, we shall first deduce part (2) of the theorem by an anticyclotomic analogue of the argument in [SU14, Thm. 3.30]. Let $I^{\text{cyc}} \subseteq \Lambda$ be the ideal generated by $\gamma^{\text{cyc}} - 1$, let $X := Ch_\Lambda(X_p(K, \mathbf{A}))$ and $Y := (\mathcal{L}_p(f/K))$. Similarly as in the proof of Theorem 5.1, the divisibility in [Wan15, Thm. 6.13] yields the divisibility $X \subseteq Y$ as ideals in Λ . On the other hand, in light of (5.3) and Corollary 1.12, Theorem 5.1 implies that $X_p(K, \mathbf{A})$ is Λ -torsion and that $X = Y \pmod{I^{\text{cyc}}}$. The equality $X = Y$ as ideals in Λ thus follows from [SU14, Lemma 3.2]. This establishes the equality in part (2), and by Theorem 3.3 the equality in part (1) also follows. \square

Corollary 5.3. *Under the hypotheses of Theorem 5.1, the module $X^{\pm, \pm}(K, \mathbf{A})$ is Λ -torsion, and*

$$Ch_\Lambda(X^{\pm, \pm}(K, \mathbf{A})) = (L_p^{\pm, \pm}(f/K))$$

as ideals in Λ .

Proof. In light of Theorem 3.3, the result of Theorem 5.2 implies that $X^{\pm, \pm}(K, \mathbf{A})$ is Λ -torsion, and the divisibility

$$(5.4) \quad Ch_\Lambda(X^{\pm, \pm}(K, \mathbf{A})) \subseteq (L_p^{\pm, \pm}(f/K))$$

as ideals in Λ . Similarly as in the proof of Theorem 5.2, we will deduce from this that equality holds in (5.4) by an appropriate application of [SU14, Lemma 3.2]. Set

$$X := Ch_\Lambda(X^{\pm, \pm}(K, \mathbf{A})), \quad Y := (L_p^{\pm, \pm}(f/K)),$$

and let I^{ac} be the kernel of the canonical projection $\Lambda \twoheadrightarrow \Lambda^{\text{cyc}}$. By the control theorem of [Wan15, Prop. 8.7] and [SU14, Lemma 3.6], the divisibility in [Kob03, Thm. 4.1] (applied to both E and its quadratic twist $E^{(K)}$) yields the divisibility

$$(5.5) \quad (X \bmod I^{\text{ac}}) \supseteq (\mathcal{L}_p^{\pm, \pm}(f/K))$$

as ideals in Λ^{cyc} , where

$$\mathcal{L}_p^{\pm, \pm}(f/K) := \mathcal{L}_p^\pm(E/\mathbf{Q}) \cdot \mathcal{L}_p(E^{(K)}/\mathbf{Q})$$

with $\mathcal{L}_p^\pm(E/\mathbf{Q})$ as in [Kob03, §3]. Thus it remains to compare the periods $\Omega_E^+ \cdot \Omega_E^-$ used in the construction of $\mathcal{L}_p^{\pm, \pm}(f/K)$ with the Petersson norm $\langle f, f \rangle_M$ used in the construction of $L_p^{\pm, \pm}(f/K)$ in Theorem 1.4. Since we may clearly ignore p -adic units, by [SZ14, Lemma 9.5] we are reduced to comparing the canonical period Ω_f^{can} of f with $\langle f, f \rangle_M$.

The following comparison may be well-known to experts, but we provide the details for the convenience of the reader. Letting D_{FL} denote the Fontaine–Laffaille functor, we have

$$D_{\text{FL}}(H_{\text{et}}^1(X_0(N), \mathbf{Z}_p)) = H^1(X_0(N), \Omega_{X_0(N)/\mathbf{Z}_p}^\bullet)$$

(see [LLZ14, §6.10]). The Galois representation $T \simeq T_f^*$ is given by $H_{\text{et}}^1(X_0(N), \mathbf{Z}_p)[\lambda_f]$, where $[\lambda_f]$ denotes the maximal submodule of $H_{\text{et}}^1(X_0(N), \mathbf{Z}_p)$ on which the Hecke algebra $\mathbb{T}_0(N)$ acts with the same eigenvalues as in f , and hence

$$D_{\text{FL}}(T) = H^1(X_0(N), \Omega_{X_0(N)/\mathbf{Z}_p}^\bullet)[\lambda_f].$$

On the other hand, we have an exact sequence of localized Hecke modules

$$H^0(X_0(N), \Omega_{X_0(N)/\mathbf{Z}_p}^1)_{\mathfrak{m}_f} \hookrightarrow H^1(X_0(N), \Omega_{X_0(N)/\mathbf{Z}_p}^\bullet)_{\mathfrak{m}_f} \twoheadrightarrow H^1(X_0(N), \mathcal{O}_{X_0(N)})_{\mathfrak{m}_f}.$$

The last term is free of rank one over $\mathbb{T}_0(N)_{\mathfrak{m}_f}$, while the first is isomorphic to $S(X_0(N), \mathbf{Z}_p)_{\mathfrak{m}_f}$. Unravelling the definitions, we see that the class $\eta_f \in H^1(X_0(N), \mathbf{Z}_p)_{\mathfrak{m}_f} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ constructed in [LLZ14, §6.10] corresponds to the projector to the f -component under the identification of $H^1(X_0(N), \mathbf{Z}_p)_{\mathfrak{m}_f} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ with $\mathbb{T}_0(N)_{\mathfrak{m}_f}$, while

$$D_{\text{FL}}(T)/\text{Fil}^0 D_{\text{FL}}(T) \simeq H^1(X_0(N), \mathcal{O}_{X_0(N)})_{\mathfrak{m}_f}[\lambda_f].$$

Thus the ratio of a generator of $D_{\text{FL}}(T)/\text{Fil}^0 D_{\text{FL}}(T)$ over η_f is, by definition, the congruence number c_f of f . Since $\langle f, f \rangle_M$ divided by c_f is Ω_f^{can} , this shows that (5.4) and (5.5) yield the equality $X = Y \pmod{I^{\text{ac}}}$, and so the equality in Corollary 5.3 follows from the divisibility (5.4) by virtue of [SU14, Lemma 3.2]. \square

5.2. Λ -adic Gross–Zagier formula. In this section we conclude the proof of Theorem B in the Introduction. Recall the p -adic height pairings $\langle \cdot, \cdot \rangle_{K_n^{\text{ac}}}^{\text{cyc}}$ on the plus/minus Selmer groups $\text{Sel}^{\pm, \pm}(K_n^{\text{ac}}, T)$ introduced in Theorem 3.6, and define the Λ^{ac} -adic height pairing

$$(5.6) \quad \langle \cdot, \cdot \rangle_{K_{\infty}^{\text{ac}}}^{\text{cyc}} : \text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}}) \otimes_{\Lambda^{\text{ac}}} \text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})^{\iota} \longrightarrow \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathcal{J}$$

by the formula

$$\langle a_{\infty}, b_{\infty} \rangle_{K_{\infty}^{\text{ac}}}^{\text{cyc}} = \varprojlim_n \sum_{\sigma \in \text{Gal}(K_n^{\text{ac}}/K)} \langle a_n, b_n^{\sigma} \rangle_{K_n^{\text{ac}}}^{\text{cyc}} \cdot \sigma.$$

Finally, define the cyclotomic Λ^{ac} -adic regulator $\mathcal{R}_{\text{cyc}}^{\pm} \subseteq \Lambda^{\text{ac}}$ to be the characteristic ideal of the cokernel of (5.6).

Theorem 5.4. *Let $\mathcal{X}_{\text{tors}}^{\pm}$ be the characteristic ideal of $X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}$. Then*

$$\mathcal{R}_{\text{cyc}}^{\pm} \cdot \mathcal{X}_{\text{tors}}^{\pm} = (L_{p,1}^{\pm, \pm}(f/K))$$

as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. The height formula of Theorem 3.6 and Lemma 3.5 immediately yield the equality

$$(5.7) \quad \mathcal{R}_{\text{cyc}}^{\pm} \cdot Ch_{\Lambda^{\text{ac}}} \left(\frac{\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathcal{BF}_{\text{ac}}^{\pm}} \right) = \mathcal{U}_{\text{ac}} \cdot \mathcal{H}_{\text{ac}}^{\pm} \cdot (L_{p,1}^{\pm, \pm}(f/K)) \cdot \eta^{\iota}$$

as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, where $\mathcal{U}_{\text{ac}} = (u_0)$ and $\mathcal{H}_{\text{ac}}^{\pm} = (h_0^{\pm})$ in the notations of §3.3, and

$$\eta := Ch_{\Lambda^{\text{ac}}} \left(\frac{H_{\pm}^1(K_{\mathbf{p}}, \mathbf{T}^{\text{ac}})}{\text{loc}_{\mathbf{p}}(\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}}))} \right).$$

By Theorem 4.6, we see that η (and therefore η^{ι}) is nonzero, while the nonvanishing of h_0^{\pm} follows from the construction (see Remark 3.2). On the other hand, from global Poitou–Tate duality we have the exact sequence

$$(5.8) \quad 0 \longrightarrow \frac{H_{\pm}^1(K_{\mathbf{p}}, \mathbf{T}^{\text{ac}})}{\text{loc}_{\mathbf{p}}(\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}}))} \longrightarrow X^{\text{rel}, \pm}(K, \mathbf{A}^{\text{ac}}) \longrightarrow X^{\pm, \pm}(K, \mathbf{A}^{\text{ac}}) \longrightarrow 0.$$

Taking Λ^{ac} -torsion in (5.8) and applying Lemma 2.7(3) we obtain the equality

$$(5.9) \quad Ch_{\Lambda^{\text{ac}}}(X^{\pm, \text{str}}(K, \mathbf{A}^{\text{ac}})) = \mathcal{X}_{\text{tors}}^{\pm} \cdot \eta^{\iota}$$

as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Combined with Corollary 3.4 (which applies thanks to Theorem 5.1) and Lemma 4.10, equation (5.9) implies that

$$(5.10) \quad Ch_{\Lambda^{\text{ac}}} \left(\frac{\text{Sel}^{\pm, \pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathcal{BF}_{\text{ac}}^{\pm}} \right) = \mathcal{H}_{\text{ac}}^{\pm} \cdot \mathcal{X}_{\text{tors}}^{\pm} \cdot \eta^{\iota}$$

as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. Since the ideal \mathcal{U}_{ac} is invertible by the combination of Theorem 3.3 and Corollary 5.3, substituting (5.10) into (5.7), the result follows. \square

Note that the formula of Theorem 5.4 has the shape of a Λ^{ac} -adic analogue of the Birch and Swinnerton-Dyer conjecture. Moreover, by the Heegner point main conjecture of Theorem 5.1, it is essentially equivalent to the following Λ -adic Gross–Zagier formula.

Theorem 5.5. *Under the hypotheses in Theorem 5.1, we have*

$$(L_{p,1}^{\pm, \pm}(f/K)) = (\langle \mathbf{z}^{\pm}, \mathbf{z}^{\pm} \rangle_{K_{\infty}^{\text{ac}}}^{\text{cyc}})$$

as ideals in $\Lambda^{\text{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. Combining Theorem 5.1 and Theorem 5.4, we obtain

$$\begin{aligned} (L_{p,1}^{\pm,\pm}(f/K)) &= \mathcal{R}_{\text{cyc}}^{\pm} \cdot Ch_{\Lambda^{\text{ac}}}(X^{\pm,\pm}(K, \mathbf{A}^{\text{ac}})_{\text{tors}}) \\ &= \mathcal{R}_{\text{cyc}}^{\pm} \cdot Ch_{\Lambda^{\text{ac}}}\left(\frac{\text{Sel}^{\pm,\pm}(K, \mathbf{T}^{\text{ac}})}{\Lambda^{\text{ac}} \cdot \mathbf{z}^{\pm}}\right)^2 \\ &= (\langle \mathbf{z}^{\pm}, \mathbf{z}^{\pm} \rangle_{K_{\infty}^{\text{ac}}}^{\text{cyc}}). \end{aligned}$$

□

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